ELEMENTS

K

OF

ALGEBRA.

FOR THE USE OF

STUDENTS IN UNIVERSITIES.

THIRD EDITION.

TO WHICH IS ADDED AN

APPENDIX.

EDINBURGH:
PRINTED FOR W. CREECH AND C. ELLIOT,
AND SOLD BY
MESS. ROBINSONS, Paternoſter-row, C. ELLIOT
AND T. KAY, N°. 332. LONDON.

M,DCC,LXXXIX.

BRITISH MUSEUM

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INTRO-

Aberdeen, April 1. 1778.

ADVERTISEMENT.

The following Syllabus was drawn up for the Uſe of Students who were to attend the Lectures of the Author, and is not intended to ſuperſede the peruſal of a more complete Syſtem of Algebra.

INTRODUCTION.

QUANTITY which can be meaſured, and is the object of Mathematics, is of two kinds, *Number* and *Extenſion.* The former is treated of in *Arithmetic*, the latter in *Geometry*.

Numbers are ranged in a *Scale*, by the continued repetition of ſome one number, which is called the *Root ;* and, in conſequence of this order, they are conveniently expreſſed in words, and denoted by characters. The operations of arithmetic are eaſily derived from the eſtabliſhed method of notation, and the moſt ſimple reaſonings concerning the relations of magnitude.

A Inveſ-

Inveſtigations by the common arithmetic are greatly limited, from the want of characters to expreſs the quantities that are unknown, and their different relations to one another, and to ſuch as are known. Hence letters, and other convenient ſymbols, have been introduced to ſupply this defect; and thus gradually has ariſen the ſcience of *Algebra*, properly called UNIVERSAL ARITHMETIC.

In the common arithmetic, too, the given numbers diſappear in the courſe of the operation, ſo that general rules can ſeldom be derived from it; but, in algebra, the known quantities, as well as the unknown, may be expreſſed by letters, which, through the whole operation, retain their original form; and hence may be deduced, not only general canons for like caſes, but the dependence of the ſeveral quantities concerned, and likewiſe the determination of a problem, without exhibiting which, it is not completely reſolved. This general manner of expreſſing quantities alſo, and the general reaſonings concerning their connections, which may be founded on it, have rendered this ſcience not leſs uſeful in the demonſtration of theorems, than in the reſolution of problems.

If geometrical quantities be ſuppoſed to be divided into equal parts, their relations, in reſpect of magnitude, or their proportions, may be expreſſed by numbers; one of theſe equal parts being denoted by the unit. Arithmetic, however, is uſed in expreſſing only the concluſions of geometrical propoſitions; and it is by algebra that the bounds and application of geometry have been of late ſo far extended.

The proper objects of mathematical ſcience are NUMBER and EXTENSION; but mathematical inquiries may be inſtituted alſo concerning any phyſical quantities, that are capable of being meaſured or expreſſed by numbers and extended magnitudes: And, as the application of algebra may be equally univerſal, it has been called *The ſcience of quantity in general.*

ELEMENTS

OF

ALGEBRA.

PART I.

DEFINITIONS.

I. QUANTITIES which are known, are generally repreſented by the firſt letters of the alphabet, as a, b, c, &c. and ſuch as are unknown, by the laſt letters, as x, y, z, &c.

II. The ſign $+$ (*plus*) denotes, that the quantity before which it is placed is to be added. Thus $a+b$ denotes the ſum of a and b; $3+5$ denotes the ſum of 3 and 5, or 8. When no ſign is expreſſed, $+$ is underſtood.

III. The ſign — (*minus*) denotes, that the quantity, before which it is placed, is to be ſubtracted. Thus, $a-b$ denotes the exceſs of a above b; $6-2$ is the exceſs of 6 above 2, or 4.

Note, Theſe characters + and —, from their extenſive uſe in Algebra, are called *the ſigns;* and the one is ſaid to be *oppoſite* or *contrary* to the other.

IV. Quantities which have the ſign + prefixed to them, are called *poſitive* or *affirmative;* and ſuch as have the ſign — prefixed to them are called *negative*.

V. Quantities which have the ſame ſign, either + or —, are alſo ſaid to have *like ſigns*, and thoſe which have different ſigns, are ſaid to have *unlike ſigns*. Thus, $+a$, $+b$, have like ſigns, and $+a$, $-c$, are ſaid to have unlike ſigns.

VI. The *juxtapoſition* of letters, as in the ſame word, expreſſes the product of the quantities denoted by theſe letters. Thus, ab expreſſes the product of a and b; bcd expreſſes the continued product of b, c, and *d*. The ſign × alſo expreſſes the product of any two quantities between which it is placed.

VII. A number prefixed to a letter is called a *numeral coefficient*, and expreſſes the product of the quantity by that number, or how often the quantity denoted by the letter is to be taken. When no number is prefixed, unit is underſtood.

VIII. The *quotient* of two quantities is denoted, by placing the *dividend* above a ſmall line, and the *diviſor* below it. Thus, $\frac{18}{3}$ is the quotient of 18 divided by 3, or 6; $\frac{a}{b}$ is the quotient of *a* divided by *b*. This expreſſion of a quotient is alſo called a *fraction*.

IX. A quantity is ſaid to be *ſimple* which conſiſts of one part or *Term*, as $+a$, $-abc$; and a quantity is ſaid to be *compound*, when it conſiſts of more than one term, connected by the ſigns + or —. Thus, $a+b$, $a-b+c$, are compound quantities. If there are two terms, it is called a *binomial;* if three, a *trinomial*, &c.

X. Simple quantities, or the terms of compound quantities, are ſaid to be *like*, which conſiſt of the ſame letter or letters, equally repeated. Thus, $+ab$, $-5ab$, are like quantities; but $+ab$, and $+aab$, are unlike.

XI. The equality of two quantities is expreſſed, by placing the ſign $=$ between them. Thus, $x+a=b-c$, means that the ſum of x and a is equal to the exceſs of b above c.

When quantities are conſidered abſtractly, the $+$ and $-$ denote addition and ſubtraction only, according to Def. II. III.; and the terms *poſitive* and *negative* expreſs the ſame ideas. In that caſe, a negative quantity by itſelf is unintelligible. The ſign $+$ alſo is unneceſſary before ſimple quantities, or before the leading term of a compound quantity which is not negative; though, when ſuch a quantity or term is to be added to another, $+$ muſt be placed before it, to expreſs that addition ; and hence in Def. II. it is ſaid, that + is underſtood, when no ſign is expreſſed.

In geometry, however, and in certain applications of geometry and algebra, there may be an oppoſition or contrariety in the quantities, analogous to that of addition and ſubtraction ; and the ſigns + and − may very conveniently be uſed to expreſs that contrariety. In ſuch caſes, negative quantities are underſtood to exiſt by themſelves ; and the ſame rules take place in operations into which they enter, as are uſed with regard to the negative terms of abſtract quantities.

When, therefore, in the following elementary rules, negative quantities are introduced as examples, they are to be underſtood, either as ſubtracted from poſitive quantities not expreſſed ; or, as having ſuch an oppoſition as has now been mentioned, to other quantities marked with the ſign.

Of Axioms.

ALGEBRAICAL reaſonings are founded on a ſet of firſt principles, not leſs evident than the axioms of Geometry. They are, indeed, ſo ſimple, that it is unneceſſary, in this place, either to enumerate, or to illuſtrate them. Beſides, a general axiom is not more evident than any particular propoſition comprehended in it.

CHAP. I.

Fundamental Operations.

THE fundamental operations in Algebra are the ſame as in common Arithmetic, ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION ; and from the various combinations of theſe four, all the others are derived.

PROB. I. *To add Quantities.*

Simple quantities, or the terms of compound quantities, to be added together, may be, *like with like ſigns*, *like with unlike ſigns*, or they may be *unlike.*

CASE I. To add terms that are like, and have like ſigns.

Rule. *Add together the coefficients, to their ſum prefix the common ſign, and ſubjoin the common letter or letters.*

EXAMPLES.

$$\begin{array}{lll} \text{To} & 5ab & 3aa-ab \\ \text{Add} & 4ab & 7aa-2ab \\ \cline{2-2} & & 4aa-5ab \\ \text{Sum} & 9ab & \\ \cline{3-3} & & 14aa-8ab \end{array}$$

Case II. To add Terms that are like, but have unlike ſigns.

Rule. *Subtract the leſs coefficient from the greater; prefix the ſign of the greater to the remainder, and ſubjoin the common letter or letters.*

EXAMPLES.

$$\begin{array}{r} -4a \\ +7a \\ \hline +3a \end{array} \qquad \begin{array}{r} +7bc \\ -3bc \\ +\ bc \\ \hline +5bc \end{array} \qquad \begin{array}{r} -5ab \\ +2ab \\ +3ab \\ \hline 0 \end{array}$$

Case III. To add Terms that are unlike.

Rule. *Set them all down, one after another, with their ſigns and coefficients prefixed.*

EXAMPLE.

$$\begin{array}{c} 2a+3b \\ -5c+8 \\ \hline 2a+3b-5c+8 \end{array}$$

Compound quantities are added together, *by uniting the ſeveral Terms, of which they conſiſt, by the preceding rules.*

EXAMPLE.

$$\text{The ſum of}\;\left\{\begin{array}{l}5ab-3xy-12cd\\7xy-ab+15\\9cd-4xy-mn\end{array}\right.$$

$$\text{is}\quad 4ab-3cd+15-mn$$

The rule for caſe III. may be conſidered as the general rule for adding all Algebraical Quantities whatſoever, and, by the rules in the two preceding caſes, the like Terms in the quantities to be added may be united, ſo as to render the expreſſion of the ſum more ſimple.

PROB. II. *To ſubtract Quantities.*

General Rule. *Change the ſigns of the quantity to be ſubtracted into the contrary ſigns, and then add it, ſo changed, to the quantity from which it was to be ſubtracted; (by Prob. I.) the ſum ariſing by this addition, is the remainder.*

EXAM-

EXAMPLES.

From	$+5a$	$7ab-16bc$
Subtract	$+3a$	$3ab+mb$
Rem.	$+2a$	$4ab-16bc-mb$

From	$5a-7b+9c+8$
Subtract	$2a-4b+9c-d$
Rem.	$3a-3b*+8+d$

When a poſitive quantity is to be ſubtracted, the rule is obvious from Def. 3.: In order to ſhew it, when the negative part of a quantity is to be ſubtracted, let $c-d$ be ſubtracted from a, the remainder, according to the rule, is $a-c+d$. For, if c is ſubtracted from a, the remainder is $a-c$, (by Def. 3.); but this is too ſmall, becauſe c is ſubtracted inſtead of $c-d$, which is leſs than it by d; the remainder therefore is too ſmall by d; and d being added, it is $a-c+d$, according to the rule.

Otherwiſe, If the quantity d be added to theſe two quantities a, and $c-d$, the difference will continue the ſame; that is, the exceſs of a above $c-d$, is equal to the exceſs of $a+d$ above $c-d+d$; that is, to the exceſs of $a+d$ above c, which plainly is

$a+d-c$, and is therefore the remainder required.

PROB. III. *To multiply quantities.*

General Rule for the ſigns. *When the ſigns of the two terms to be multiplied are like, the ſign of the product is* + ; *but, when the ſigns are unlike, the ſign of the product is* —.

CASE I. To multiply two terms.

Rule. *Find the ſign of the product by the general rule; after it place the product of the numeral coefficients, and then ſet down all the letters one after another, as in one word.*

Mult.	$+a$	$+5b$	$-5ax$
By	$+b$	$-3c$	$-7ab$
	$+ab$	$-15bc$	$+35aabx$

The reaſon of this rule is derived from Def. 6. and from the nature of multiplication, which is a repeated addition of one of the quantities to be multiplied as often as there are units in the other. Hence alſo the letters in two terms multiplied together may be placed in any order, and therefore the order of the alphabet is generally preferred.

CASE II. To multiply compound quantities.

Rule. *Multiply every term of the multiplicand by all the terms of the multiplier, one after another, according to the preceding rule, and then collect all the products into one ſum; that ſum is the product required.*

EXAMPLES.

$$\begin{array}{lll}
\text{Mult.} & 2a+3b & m+x\\
\text{By} & 3ax-4by & m-x\\
\hline
 & 6aax+9abx & mm+mx\\
 & \qquad -8aby-12bby & -mx-xx\\
\hline
\text{Prod.} & 6aax+9abx-8aby-12bby & mm\ast-xx
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & a-b\\
\text{By} & c-d\\
\hline
 & ac-cb\\
 & \quad -ad+db\\
\hline
\text{Prod.} & ac-cb-ad+db
\end{array}$$

Of

Of the General Rule for the Signs.

The reaſon of that rule will appear, by proving it, as applied to the laſt mentioned example of $a-b$ multiplied by $c-d$, in which every caſe of it occurs.

Since multiplication is a repeated addition of the multiplicand, as often as there are units in the multiplier; hence, if $a-b$ is to be multiplied by c, $a-b$ muſt be added to itſelf, as often as there are units in c, and the product therefore muſt be $ca-cb$, (Prob. I.).

But this product is too great; for $a-b$ is to be multiplied, not by c, but by $c-d$ only, which is the exceſs of c above d; d times $a-b$, therefore, or $da-db$, has been taken too much; hence, this quantity muſt be ſubtracted from the former part of the product, and the remainder, which (by prob. II.) is $ca-cb-da+db$, will be the true product required.

Def. XII. When ſeveral Quantities are multiplied together, any of them is called a *factor* of the product.

XIII. The products ariſing from the continual multiplication of the ſame quantity, are called the *Powers* of that quantity, which is the *Root*. Thus, aa, aaa, $aaaa$, &c. are *powers* of the *root* a.

XIV. Theſe powers are expreſſed, by placing above the root, to the right hand, a figure, denoting how often the root is repeated. This figure is called an *index* or *exponent*, and from it the power is denominated. Thus,

a	is called the	1ſt	*Power* of the *Root* a, and is otherwiſe expreſſed, by	a^1 or a
aa		2d		a^2
aaa		3d		a^3
$aaaa$		4th		a^4, &c.

The 2d and 3d powers are generally called the *Square* and *Cube*; and the 4th, 5th, and 6th, are alſo ſometimes reſpectively called the *Biquadrate*, *Surſolid*, and *Cubocube*.

Cor. Powers of the ſame root are multiplied by adding their exponents. Thus, $a^2\times a^3=a^5$, or $aaa\times aa=aaaaa$, $b^3\times b=b^4$.

SCHO-

SCHOLIUM.

Sometimes it is convenient to expreſs the multiplication of quantities, by ſetting them down with the ſign × between them, without performing the operation according to the preceding rules ; thus, $a^2 \times b$ is written inſtead of a^2b ; and $\overline{a - b} \times \overline{c - d}$, expreſſes the product of $a - b$, multiplied by $c - d$.

Def. XV. A *Vinculum* is a line drawn over any number of terms of a compound quantity, to denote thoſe which are underſtood to be affected by the particular ſign connected with it.

Thus, in the laſt example, it ſhews, that the terms $+a$ and $-b$, and alſo c and $-d$ are all affected by the ſign ×. Without the vinculum, the expreſſion $a - b \times c - d$, would mean the exceſs of a above bc and d ; and $\overline{a - b} \times c - d$ would mean the exceſs of the product of $a - b$ by c, above d. Thus alſo, $\overline{a + b)}^2$, expreſſes the ſecond power of $a + b$, or the product of that quantity multiplied by itſelf, ; whereas $a + b^2$ would expreſs only the ſum of a and b^2; and ſo of others. By ſome writers a *parentheſis* ( ) is uſed as a vinculum, and $(a+b)^2$ is the ſame thing as $\overline{a+b})^2$.

PROB. IV. *To divide quantities.*

General rule for the ſigns. *If the ſigns of diviſor and dividend are like, the ſign of the quotient is* + ; *if they are unlike, the ſign of the quotient is* —.

This rule is eaſily deduced from that given in Prob. 3. ; for, from the nature of diviſion, the quotient muſt be ſuch a quantity as, multiplied by the diviſor, ſhall produce the dividend, with its proper ſign.

From Def. 8. the Quotient of any two Quantities may be expreſſed, by placing the dividend above a line, and the diviſor below it. But a quotient may often be expreſſed in a more ſimple and convenient form, as will appear from the following diſtinction of the Caſes.

Caſe I. When the diviſor is ſimple, and is a *factor* of all the terms of the Dividend. This is eaſily diſcovered by inſpection ; for then the coefficient of the diviſor meaſures that of all the Terms of the dividend, and all the letters of the diviſor are found in every Term of the Dividend.

Rule. *The letter or letters in the diviſor are to be expunged out of each term in the dividend, and the coefficients of each term to be divided by the coefficient of the diviſor; the quantity reſulting is the quotient.*

Ex. $a)\,ab(b.$ $\quad 2aab)\,6a^3bc—4a^2bdm\,(3ac—2dm.$

The reaſon of this is evident from the nature of diviſion, and from def. 6.

Note. It is obvious from corollary to prob. 3. that powers of the ſame root are divided by ſubtracting their exponents.

Thus, $a^2)a^3(a.$ $\quad a^3)a^7(a^4.$ $\quad$ alſo $a^2b)a^3b^6(ab^5.$

Case II. When the diviſor is ſimple, but not a factor of the dividend.

Rule. *The quotient is expreſſed by a fraction, according to def.* 8. viz. *by placing the*

the dividend above a line, and the diviſor below it.

Thus, the quotient of $3ab^2$ divided by $2mbc$ is the fraction $\frac{3ab^2}{2mbc}$.

Such expreſſions of quotients may often be reduced to a more ſimple form, as ſhall be explained in the ſecond part of this chapter.

Case III. When the diviſor is compound.

RULE.

1. *The terms of the dividend are to be ranged according to the powers of ſome one of its letters; and thoſe of the diviſor, according to the powers of the ſame letter.*

Thus, if $a^2+2ab+b^2$ is the dividend, and $a+b$ the diviſor, they are ranged according to the powers of a.

2. *The firſt term of the dividend is to be divided by the firſt term of the diviſor, (obſerving*

ſerving the general rule for the ſigns ;) and this quotient being ſet down as a part of the quotient wanted, is to be multiplied by the whole diviſor, and the product ſubtracted from the dividend. If nothing remain, the diviſion is finiſhed : the remainder, when there is any, is a new dividend.

Thus, in the preceding example, a^2 divided by a, gives a, which is the firſt part of the quotient wanted : and the product of this part by the whole diviſor $a+b$, viz. a^2+ab being ſubtracted from the given dividend, there remains in this example $ab+b^2$.

3. *Divide the firſt term of this new dividend by the firſt term of the diviſor as before, and join the quotient to the part already found, with its proper ſign : then multiply the whole diviſor by this part of the quotient, and ſubtract the product from the new dividend : and thus the operation is to be continued, till no remainder is left, or till it appear that there will always be a remainder.*

Thus, in the preceding example, $+ab$, the firſt term of the new dividend divided by a, gives b; the product of which, multiplied by $a+b$, being ſubtracted from $ab+b^2$, nothing remains, and $a+b$ is the true quotient.

The entire operation is as follows.

$$\begin{array}{l}
a+b)a^2+2ab+b^2(a+b \\
\quad\quad\; a^2+ab \\
\hline
\quad\quad\quad\quad ab+b^2 \\
\quad\quad\quad\quad ab+b^2 \\
\hline
\end{array}$$

$$\begin{array}{l}
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; * \; * \\
3a-b)3a^3-12a^2-a^2b+10ab-2b^2(a^2-4a+2b \\
\quad\quad\;\; 3a^3 \quad\quad\quad\; -a^2b \\
\hline
\quad\quad\quad -12a^2 \quad\quad +10ab \\
\quad\quad\quad -12a^2 \quad\quad +\;4ab \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad +6ab-2b^2 \\
\quad\quad\quad\quad\quad\quad\quad\quad +6ab-2b^2 \\
\hline
\end{array}$$

$$\begin{array}{l}
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * \quad\; * \\
1-a)\,1 \quad (1+a+a^2+a^3, \text{ \&c.} \\
\quad\quad\; 1-a \\
\hline
\quad\quad\quad\; +a \\
\quad\quad\quad\; +a-a^2 \\
\hline
\quad\quad\quad\quad\; +a^2 \\
\quad\quad\quad\quad\; +a^2-a^3 \\
\hline
\quad\quad\quad\quad\quad\quad +a^3, \text{ \&c.}
\end{array}$$

It often happens, as in the laſt example, that there is ſtill a remainder from which the operation may be continued without end. This expreſſion of a quotient is called *an infinite ſeries*; the nature of which ſhall be conſidered afterwards. By comparing a few of the firſt terms, the *law* of the ſeries may be diſcovered, by which, without any more diviſion, it may be continued to any number of terms wanted.

Of the General Rule.

The reaſon of the different parts of this rule is evident; for, in the courſe of the operation, all the terms of the quotient obtained by it are multiplied by all the terms of the diviſor, and the products are ſucceſſively ſubtracted from the dividend, till nothing remain: that, therefore, from the nature of diviſion, muſt be the true quotient.

Note. The ſign ÷ is ſometimes uſed to expreſs the quotient of two quantities, between which it is placed: Thus, $\overline{a^2+x^2}\div\overline{a+x}$, expreſſes the quotient of a^2+x^2 divided by $a+x$.

CHAP. I.

PART II.

OF FRACTIONS.

DEFINITIONS.

I. WHEN a quotient is expreſſed by a fraction, the diviſion above the line is called the *numerator* ; and the diviſor below it is called the *denominator*.

II. If the numerator is leſs than the denominator, it is called a *proper fraction*.

III. If the numerator is not leſs than the denominator, it is called an *improper fraction*.

IV. If one part of a quantity is an integer, and the other a fraction, it is called a *mixt quantity*.

V. The *reciprocal* of a fraction, is a fraction whoſe numerator is the denominator of the other; and whoſe denominator is the numerator of the other. The reciprocal of an integer is the quotient of 1 divided by that integer. Thus, $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$; and $\frac{1}{m}$ is the reciprocal of m.

The diſtinctions in Def. II. III. IV. properly belong to common arithmetic, from which they are borrowed, and are ſcarcely uſed in Algebra.

The operations concerning fractions are founded on the following propoſition:

If the diviſor and dividend be either both multiplied, or both divided, by the ſame quantity, the quotient is the ſame; or, if both the numerator and denominator of the fraction be either multiplied or divided by the ſame

ſame quantity, the value of that fraction is the ſame.

Thus, let $\frac{a}{b} = c$, then $\frac{ma}{mb} = c$. For, from the nature of diviſion, if the quotient $\frac{a}{b}$ $(= c)$ be multiplied by the diviſor b, the product muſt be the dividend a. Hence $\left(\frac{a}{b} \times b = \right) bc = a$, and likewiſe $ma = mbc$, and dividing both by mb, $\frac{ma}{mb} = c$. Converſely, if $\frac{ma}{mb} = c$, then alſo $\frac{a}{b} = c$.

Cor. 1. Hence a fraction may be reduced to another of the ſame value, but of a more ſimple form, by dividing both numerator and denominator by any common meaſure.

Thus, $\frac{30ax - 54ay}{12ab} = \frac{5x - 9y}{2b}$.

$$\frac{8ab + 6ac}{4a^2} = \frac{4b + 3c}{2a}.$$

Cor. 2. A fraction is multiplied by any integer, by multiplying the numerator, or dividing the denominator by that integer: and converſely, a fraction is divided by any integer, by dividing the numerator, or multiplying the denominator by that integer.

Prob. I. To find the greateſt common meaſure of two quantities.

1. Of pure numbers.

Rule. *Divide the greater by the leſs: and, if there is no remainder, the leſs is the greateſt common meaſure required. If there is a remainder, divide the laſt diviſor by it; and thus proceed continually dividing the laſt diviſor by its remainder, till no remainder is left, and the laſt diviſor is the greateſt common meaſure required.*

The greateſt common meaſure of 45 and 63 is 9; the greateſt common meaſure of 187 and 391 is 17. Thus,

```
45)63(1              207)391(2
   45                    374
   —                  ———————
   18(45(2                17)187(11
      36                     187
      —                     ————
      9)18(2                  *
        18
        —
        0
```

Fro

From the nature of this operation, it is plain, that it may always be continued, till there be no remainder. The rule depends on the two following principles.

1. A quantity which meaſures both diviſor and remainder, muſt meaſure the dividend.

2. A quantity which meaſures both diviſor and dividend, muſt alſo meaſure the remainder.

For a quantity which meaſures two other quantities, muſt alſo meaſure both their ſum and difference; and, from the nature of diviſion, the dividend conſiſts of the diviſor repeated a certain number of times, together with the remainder. By the firſt it appears, that the number found by this rule is a common meaſure; and, by the ſecond, it is plain there can be no greater common meaſure; for, if there were, it muſt neceſſarily meaſure the quantity already found leſs than itſelf, which is abſurd.

When the greateſt common meaſure of algebraical quantities is required, if either of them be ſimple, any common ſimple diviſor is eaſily found by inſpection. If they are both compound, any common ſimple diviſor may alſo be found by inſpection. But, when the greateſt compound diviſor is wanted, the preceding rule is to be applied; only,

2. *The ſimple diviſors of each of the quantities are to be taken out, the remainders in the ſeveral operations are alſo to be divided by their ſimple diviſors, and the quantities are always to be ranged according to the powers of the ſame letter.*

The ſimple diviſors in the given quantities, or in the remainders, do not affect a compound diviſor which is wanted; and hence alſo, to make the diviſion ſucceed, any of the dividends may be multiplied by a ſimple quantity. Beſides the ſimple diviſors in the remainders not being found in the diviſors from which they ariſe, can make no part of the common meaſure ſought; and for the ſame reaſon, if in ſuch a remainder, there be any compound diviſor which does not meaſure the diviſor from which it proceeds, it may be taken out.

EXAMPLES.

$$a^2 - b^2\ )\ a^2 - 2ab + b^2\ (1$$
$$a^2 - b^2$$
$$\overline{}$$

$-2ab+2b^2$ Remainder, which, divided by $-2b$ is $a-b$) a^2-b^2 $(a+b$

$$a^2 - b^2$$
$$\overline{}$$

* *

If the quantities given are, $8a^2b^2-10ab^3+2b^4$, and $9a^4b-9a^3b^2+3a^2b^3-3ab^4$. The ſimple diviſors being taken out, viz. $2b^2$ out of the firſt, it becomes $4a^2-5ab+b^2$, and $3ab$ out of the ſecond, it is $3a^3-3a^2b+ab^2-b^3$. As the latter is to be divided by the former, it muſt be multiplied by 4, to make the operation ſucceed, and then it is as follows :

$$4a^2-5ab^2)\ 12a^3-12a^2b+4ab^2-4b^3\ (3a$$
$$12a^3-15a^2b+3ab^2$$
$$\overline{}$$
$$3a^2b+ab^2-4b^3$$

This remainder is to be divided by b, and the new dividend multipled by 3, to make the diviſion proceed. Thus,

$$\begin{array}{r} 3a^2+ab-4b^2)\ 12a^2-15ab+3b^2\ (4 \\ 12a^2+4ab-16b^2 \\ \hline -19ab+19b^2 \end{array}$$

and this remainder, divided by $-19b$, gives $a-b$, which being made a diviſor, divides $3a^2+ab-4b^2$, without a remainder, and therefore $a-b$ is the greateſt compound diviſor: but there is a ſimple diviſor b, and therefore $\overline{a-b}\times b$ is the greateſt common meaſure required.

Prob. II. To reduce a fraction to its loweſt terms.

Role. *Divide both numerator and denominator by their greateſt common meaſure, which may be found by prob.* 1.

Thus, $\frac{75abc}{125bcx}=\frac{3a}{5x}$, $25bc$ being the greateſt common meaſure. $\frac{a^4-b^4}{a^5-a^3b^2}=\frac{a^2+b^2}{a^3}$. Alſo, $\frac{9a^4b-9a^3b^2+3a^2b^3-3ab^4}{8a^2b^2-10ab^3+2b^4}=\frac{9a^3+3ab^2}{8ab-2b^2}$, the greateſt common meaſure being $\overline{a-b}\times b$, by prob. 1.

Prob. III. To reduce an integer to the form of a fraction.

Rule. *Multiply the given integer by any quantity for a numerator, and ſet that quantity under the product, for a denominator.*

Thus, $a=\frac{ma}{m}$, $a+b=\frac{a^2-b^2}{a-b}$.

Cor. Hence, in the following operations concerning fractions, an integer may be introduced; for, by this problem, it may be reduced to the form of a fraction. The denominator of an integer is generally made 1.

Prob. IV. To reduce fractions with different denominators to fractions of equal value, that ſhall have the ſame denominator.

Rule. *Multiply each numerator, ſeparately taken, into all the denominators but its own, and the products ſhall give the new numerators. Then multiply all the denominators into one another, and the product ſhall give the common denominator.*

Example. Let the fractions be $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, they are reſpectively equal to $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$, $\frac{bde}{bdf}$.

The reaſon of the operation appears from the preceding propoſition; for the numerator and denominator of each fraction are multiplied by the ſame quantities; and the value of the fractions therefore is the ſame.

Prob. V. To add and ſubtract fractions.

Rule. *Reduce them to a common denominator, then add or ſubtract the numerators; and the ſum or difference ſet over the common denominator is the ſum or remainder required.*

Ex. Add together $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, the ſum is $\frac{adf+cbf+bde}{bdf}$.

From $\frac{a}{b}$ ſubtract $\frac{c}{d}$, the difference is $\frac{ad-bc}{bd}$.

From the nature of diviſion it is evident, that, when ſeveral quantities are to be divided by the ſame diviſor, the ſum of the quotients is the ſame with the quotient of the ſum of the quantities divided by that common diviſor.

In like manner, the difference of two fractions having the ſame denominator, is equal to the difference of the numerators divided by that common denominator.

Cor. 1. By Cor. Prob. 3. integers may be reduced to the form of fractions, and hence integers and fractions may be added and ſubtracted by this rule. Hence alſo what is called a mixt quantity may be reduced into the form of a fraction by bringing the integral part into the form of a fraction, with the ſame denominator as the fractional part, and adding or ſubtracting the numerators according as the two parts are connected by the ſigns + or —.

Thus, $b + \frac{c}{d} = \frac{bd+c}{d}$, and $a - \frac{a^2-b^2}{2a} = \frac{2a^2-a^2+b^2}{2a} = \frac{a^2+b^2}{2a}$.

Cor. 2. A fraction, whoſe numerator is a compound quantity, may be diſtinguiſhed into parts, by dividing the numerator into ſeveral parts, and ſetting each over the original denominator, and uniting the new fractions, (reduced if neceſſary) by the ſigns of their numerators.

Thus, $\frac{a^2-2ab+b^2}{2a}=\frac{a^2}{2a}-\frac{2ab}{2a}+\frac{b^2}{2a}=\frac{a}{2}-b+\frac{b^2}{2a}$.

Prob. V. To multiply fractions.

Rule. *Multiply their numerators into one another, to obtain the numerator of the product; and the denominators, multiplied into one another, ſhall give the denominator of the product.*

Ex. $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$. $\frac{a+b}{c}\times\frac{a-b}{d}=\frac{a^2-b^2}{cd}$.

For, if $\frac{a}{b}$ is to be multiplied by c, the product is $\frac{ca}{b}$; but if it is to be multiplied only by $\frac{c}{d}$, the former product muſt be divided by d, and it becomes $\frac{ca}{bd}$; (cor. 2. to the preceding problem).

Or let $\frac{a}{b}=m$, and $\frac{c}{d}=n$. Then $a=bm$, and $c=dn$, and $ac=bdmn$, and $(mn=)\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

Prob. VII. To divide fractions.

Rule. *Multiply the numerator of the dividend by the denominator of the diviſor; their*

their product ſhall give the numerator of the quotient. Then multiply the denominator of the dividend by the numerator of the diviſor, and their product ſhall give the denominator.

Or, *Multiply the dividend by the reciprocal of the diviſor, the product will be the quotient wanted.*

Thus, $\frac{a}{b}\Big)\,\frac{c}{d}\,\Big(\frac{bc}{ad}=\frac{c}{d}\times\frac{b}{a}$.

For, if $\frac{c}{d}$ is to be divided by a, the quotient is $\frac{c}{ad}$; but $\frac{c}{d}$ is to be divided, not by a, but by $\frac{a}{b}$; therefore the former quotient muſt be multiplied by b, and it is $\frac{bc}{da}$.

Or, let $\frac{a}{b}=m$ and $\frac{c}{d}=n$; then $a=bm$, and $c=dn$; alſo $ad=bdm$, and $bc=bdn$; therefore $\frac{bdn}{bdm}=\frac{n}{m}=\frac{bc}{ad}$.

SCHOLIUM.

By theſe problems, the four fundamental operations may be performed, when any terms of the original quantities, or of thoſe which ariſe in the courſe of the operation, are fractional.

Example, Mult. $\frac{a^2}{2x} - \frac{3ax}{2b}$

By $\frac{ab}{3x} - 4x$

Prod. $\frac{a^3b}{6x^2} - \frac{a^2}{2} - 2a^2 + \frac{6ax^2}{b}$.

$$
\begin{array}{l}
a+x)\ a^2+x^2\ (a-x+\frac{2x^2}{a}-\frac{2x^3}{a^2}\ \&c. \\
\qquad\ a^2+ax \\
\hline
\qquad -ax+x^2 \\
\qquad -ax-x^2 \\
\hline
\qquad\qquad 2x^2 \\
\qquad\qquad 2x^2+\frac{2x^3}{a} \\
\hline
\qquad\qquad\quad -\frac{2x^3}{a} \\
\qquad\qquad\quad -\frac{2x^3}{a}-\frac{2x^4}{a^2} \\
\hline
\qquad\qquad\qquad\quad +\frac{2x^4}{a}\ \&c.
\end{array}
$$

This quotient becomes a ſeries, of which the law of continuation is obvious, without any farther operation.

In ſuch caſes, when we arrive at a remainder of one term, it is commonly ſet down with the diviſor below it, after the other terms of the quotient, which then becomes a mixt quantity

Thus, the laſt quotient is alſo expreſſed by $a-x+\frac{2x^2}{a+x}$.

CHAP.

CHAP. II.

Of Proportion.

BY the preceding operations, quantities of the ſame kind may be compared together.

The relation ariſing from this compariſon is called *Ratio* or *Proportion*, and is of two kinds. If we conſider the difference of the two quantities, it is called *Arithmetical Proportion*; and, if we conſider their quotient, it is called *Geometrical Proportion*. This laſt being moſt generally uſeful, is commonly called ſimply *Proportion*.

1. *Of Arithmetical Proportion.*

Definition. When, of four quantities, the difference of the firſt and ſecond is equal to the difference of the third and fourth, the quantities are called *arithmetical proportionals*.

Cor. Three quantities may be arithmetically proportional, by ſuppoſing the two middle terms of the four to be equal.

Prop. In four quantities arithmetically proportional, the ſum of the extremes is equal to the ſum of the means.

Let the four be a, b, c, d. Therefore from Def. $a-b=c-d$; to theſe add $b+d$ and $a+d=b+c$.

Cor. 1. Of four arithmetical proportionals, any three being given, the fourth may be found.

Thus, let a, b, c, be the 1ſt, 2d, and 4th terms, and let x be the third which is ſought.

Then, by definition, $a+c=b+x$, and $x=a+c-b$.

Cor. 2. If three quantities be arithmetical proportionals, the ſum of the extremes is double of the middle term; and hence, of three ſuch proportionals, any two being given, the third may be found.

2. *Of Geometrical Proportion.*

Definition. If, of four quantities, the quotient of the firſt and ſecond is equal to the quotient of the third and fourth, theſe quantities are ſaid to be in *geometrical proportion*. They are alſo called *proportionals*.

Thus, if a, b, c, d, are the four quantities, then $\frac{a}{b} = \frac{c}{d}$, and their ratio is thus denoted, $a : b :: c : d$.

Cor. Three quantities may be geometrical proportionals, viz. by ſuppoſing the two middle terms of the four to be equal. If the quantities are a, b, c, then $\frac{a}{b} = \frac{b}{c}$, and the proportion is expreſſed thus, $a : b : c$.

Prop. I. The product of the extremes of four quantities geometrically proportional is equal to the product of the means: and converſely.

Let $a : b :: c : d$.

Then, by Def. $\frac{a}{b} = \frac{c}{d}$,

and multiplying both by bd, $ad = bc$.

If $ad = bc$, then dividing by bd, $\frac{a}{b} = \frac{c}{d}$, that is, $a : b :: c : d$.

Cor. 1. The product of the extremes of three quantities, geometrically proportional, is equal to the ſquare of the middle term.

Cor. 2. Of four quantities geometrically proportional, any three being given, the fourth may be found.

Ex. Let a, b, c, be the three firſt; to find the 4th. Let it be x, then $a : b :: c : x$, and by this propoſition,

$$ax = bc$$

and dividing both by a, $x = \frac{bc}{a}$.

This coincides with the *Rule of Three* in arithmetic, and may be conſidered as a demonſtration of it. In applying the rule to any particular caſe, it is only to be obſerved, that the quantities muſt be ſo connected, and ſo arranged, that they be proportional, according to the preceding definition.

Cor. 3. Of three geometrical proportionals, any two being given, the third may be found.

Prop. II. If four quantities be geometrically proportional, then if any equimultiples whatever be taken of the firſt and third, and alſo any equimultiples whatever of the ſecond and fourth; if the multiple of the firſt be greater than that of the ſecond, the multiple of the third will be greater than that of the fourth; and if equal, equal; and if leſs, leſs.

For, let a, b, c, d, be the four proportionals. Of the firſt and third, ma and mc may repreſent any equimultiples whatever, and alſo nb, nd, may repreſent any equimultiples of the ſecond and fourth. Since $a : b :: c : d$, $ad = bc$; and hence multiplying by mn, $mnad = mnbc$, and therefore (Conv. Prop. 1.) $ma : nb :: mc : nd$; and from the definition of proportionals, it is plain, that if ma is greater than nb, mc muſt be greater than nd; and if equal, equal: and if leſs, leſs.

Prop. III. If four quantities are proportionals, they will alſo be proportionals when taken *alternately* or *inverſely*, or by *compoſition*, or by *diviſion*, or by *converſion*. See def. 13. 14. 15. 16. 17. of book V. of Euclid, Simſon's edition.

By Prop. II. they will alſo be proportionals, according to Def. 5. book V. of Euclid; and therefore this propoſition is demonſtrated by Propoſitions 16, B, 18, 17, E, of the ſame book.

Otherwiſe algebraically,

Let $a : b :: c : d$, and therefore $ad = bc$,

Altern. $a : c :: b : d$
Invert. $b : a :: d : c$
Divid. $a-b : b :: c-d : d$
Comp. $a+b : b :: c+d : d$
Convert. $a : a-b :: c-d$

For, ſince $ad = bc$, it is obvious that in each of theſe caſes the product of the extremes is equal to the product of the means; the quantities are therefore proportionals, (prop. 1.).

Prop. IV. If four numbers be proportionals, according to Def. 5. B. V. of Euclid, they will be geometrically proportional, according to the preceding definition.

1ſt, Let the four numbers be integers, and let them be a, b, c, d. Then, if b times a and b times c be taken, and alſo a times b and a times d, ſince ba, the multiple of the firſt, is equal to ab, the multiple of the ſecond,

cond, *bc*, the multiple of the third, muſt be equal to *ad*, the multiple of the fourth. And, ſince $bc=ad$, by prop. 1. *a*, *b*, *c*, and *d*, muſt be geometrical proportionals.

2*dly*, If any of the numbers be fractional, all the four being multiplied by the denominators of the fractions, they continue proportionals, according to Def. 5. B. V. Euclid, (by Prop. IV. of that book), and the four integer quantities produced being ſuch proportionals, they will be geometrical proportionals, by the firſt part of this Prop.; and, therefore, being reduced by diviſion to their original form, they manifeſtly will remain proportionals, according to the algebraical definition.

CHAP.

C H A P. III.

Of Equations in general, and of the Solution of ſimple Equations.

DEFINITIONS.

I. AN *Equation* may in general be defined to be a propoſition aſſerting the equality of two quantities; and is expreſſed by placing the ſign = between them.

II. When a quantity ſtands alone upon one ſide of an equation, the quantities on the other ſide are ſaid to be a *value* of it. Thus, in the equation $x=b+y-d$, x ſtands alone on one ſide, and $b+y-d$ is a value of it.

III. When an unknown quantity is made to ſtand alone on each ſide of an equation, and there are only known quantities on the other; that equation is ſaid to be *reſolved*; and the value of the unknown quantity is called a *root* of the equation.

IV. Equations containing only one unknown quantity and its powers, are divided into *orders*, according to the higheſt power of the unknown quantity to be found in any of its terms.

If the higheſt power of the unknown quantity in any term be the	1ſt,	The Equation is called	*Simple,*
	2d,		*Quadrat.*
	3d,&c.		*Cubic,*&c.

But the exponents of the unknown quantities are ſuppoſed to be integers, and the equation is ſuppoſed to be cleared of fractions, in which the unknown quantity, or any of its powers, enter the denominators.

Thus, $x+a=\frac{3x-b}{c}$ is a ſimple equation; $3x-\frac{5}{2x}=12$, when cleared of the fraction by multiplying both ſides by $2x$, becomes $6x^2-5=24x$, a quadratic. $x^3-2x^4=x^6-20$ is an equation of the 6th order, &c.

As the general relations of quantity which may be treated of in Algebra, are almoſt univerſally either that of equality, or ſuch as may be reduced to that of equality, the doctrine of equations becomes one of the chief branches of the ſcience.

The moſt common and uſeful application of Algebra, is the inveſtigation of quantities that are unknown, from certain given relations to each other, and to ſuch as are known; and hence it has been called the *Analytical Art*. The equations employed for expreſſing theſe relations muſt therefore contain one or more unknown quantities, and the principal buſineſs of this art will be, the deducing equations containing only one unknown quantity, and reſolving them.

The ſolution of the different orders of equations will be ſucceſſively explained; the preliminary rules in the following ſection are uſeful in all orders, and are alone ſufficient for the ſolution of ſimple equations.

SECT. I.

Of Simple Equations, and their Reſolution.

Simple equations are reſolved by the four fundamental operations already explained; and the application of them to this purpoſe is contained in the following rules.

Rule I. *Any quantity may be tranſpoſed from one ſide of an equation to the other, by changing its ſign.*

Thus, if $3x-10=2x+5$,

Then, $3x-2x=10+5$, or $x=15$.

Thus alſo, $5x+b=a+2x$,

By tranſ. $3x=a-b$.

This rule is obvious from prob. 1. and 2; for it is equivalent to adding equal quantities to both ſides of the equation, or to ſubtracting equal quantities from both ſides.

Cor. The ſigns of all the terms of an equation

quation may be changed into the contrary ſigns, and it will continue to be true.

Rule II. *Any quantity by which the unknown quantity is multiplied may be taken away, by dividing all the other quantities of the equation by it.*

Thus, if $ax = b$

$$x = \frac{b}{a}.$$

Alſo, if $mx + nb = am$

$$x + \frac{nb}{m} = a.$$

For if equal quantities are divided by the ſame quantity, the quotients are equal.

Rule III. *If a term of an equation is fractional, its denominator may be taken away, by multiplying all the other terms by it.*

Thus, if $\frac{x}{a} = b + c$, Alſo, if $a - \frac{b}{x} = c$,

$x = ab + ac$. $ax - b = cx$,

And by tranſ. $ax - cx = b$,

And by div. $x = \frac{b}{a - c}$.

For, if all the terms of the equation are multiplied by the ſame quantity, it remains a true propoſition.

Corollary to the three laſt Rules.

If any quantity be found on both ſides of the equation, with the ſame ſign, it may be taken away from both. (Rule I.).

Alſo, if all the terms in the equation are multiplied or divided by the ſame quantity, it may be taken out of them all. (Rule II. and III.).

Ex. If $3x+a=a+b$, then $3x=b$.

If $2ax+3ab=ma+a^2$, then $2x+3b=m+a$.

If $\frac{x}{3}-\frac{4}{3}=\frac{16}{3}$, then $x-4=16$.

Any ſimple equation may be reſolved by theſe rules in the following manner: 1*ſt*, Any fraction may be taken away by R. 3. 2*dly*, All the terms including the unknown quantity may be brought to one ſide of the equation, and the known terms to the other, by R. 1. *Laſtly*, If the unknown quantity is multiplied by any known quantity, it may be made to ſtand alone by R. 2. and the equation will then be reſolved. Def. 3.

Examples of Simple Equations reſolved by theſe Rules.

I.

If $3x + 5 = x + 9$.

R. 1. $2x = 4$

R. 2. $x = \frac{4}{2} = 2$.

II.

If $5x - \frac{5x}{2} + 12 = \frac{4x}{3} + 26$

R. 1. $5x - \frac{5x}{2} - \frac{4x}{3} = 14$

R. 3. $30x - 15x - 8x = 84$

Or $7x = 84$

R. 2. $x = \frac{84}{7} = 12$.

III.

If $\frac{5}{x} + \frac{9}{4} = 16$

R. 3. $\frac{20}{x} + 9 = 64$

R. 3. $20 + 9x = 64x$

R. 1. $20 = 55x$

R. 2. $x = \frac{20}{55} = \frac{4}{11}$.

SECT.

SECT. II. *Solution of Queſtions producing ſimple Equations.*

From the reſolution of equations we obtain the reſolution of a variety of uſeful problems, both in pure mathematics and phyſics, and alſo in the practical arts founded upon theſe ſciences. In this place, we conſider the application of it to thoſe queſtions where the quantities are expreſſed by numbers, and their magnitude alone is to be conſidered.

When an equation, containing only one unknown quantity, is deduced from the queſtion by the following rules, it is ſometimes called a *Final Equation.* If it be ſimple, it may be reſolved by the preceding rules ; but, if it be of a ſuperior order, it muſt be reſolved by the rules afterwards to be explained. The examples in this chapter are ſo contrived, that the final equation may be ſimple.

The rules given in this ſection, for the ſolution of queſtions, though they contain a reference to ſimple equations only, are to be conſidered as general, and as applicable to queſtions which produce equations of any order.

GENERAL RULE.

The unknown quantities in the queſtion propoſed muſt be expreſſed by letters, and the relations of the known and unknown quantities contained in it, or the conditions of it, as they are called, muſt be expreſſed by equations. Theſe equations being reſolved by the rules of this ſcience, will give the anſwer of the queſtion.

For example, if the queſtion is concerning two numbers, they may be called x and y, and the conditions from which they are to be inveſtigated, muſt be expreſſible by equations.

Thus, if it be required that the ſum of two numbers ſought be 60, that condition is expreſſed thus, $x+y=60$.

If their difference muſt be 24, then $x-y=24$.

If their product is 1640, then $xy=1640$.

If their quotient muſt be 6, then $\frac{x}{y}=6$.

If their ratio is as 3 to 2, then $x:y::3:2$, and therefore $2x=3y$.

Theſe are ſome of the relations, which are moſt eaſily expreſſed: Many others occur, which are leſs obvious; but, as they cannot be deſcribed in particular rules, the algebraical expreſſion of them is beſt explained by examples, and muſt be acquired by experience.

A diſtinct conception of the nature of the queſtion, and of the relations of the ſeveral quantities to which it refers, will generally lead to the proper method of ſtating it, which in effect may be conſidered only as a tranſlation from common language, into that of Algebra.

Case I. When there is only one unknown quantity to be found.

Rule. *An equation involving the unknown quantity muſt be deduced from the queſtion (by the general rule). This equation being reſolved by the rules of the laſt ſection, will give the anſwer.*

It is obvious, that, when there is only one unknown quantity, there muſt be only one independent equation contained in the queſtion; for any other would be unneceſſary, and might be contradictory to the former.

EXAMPLE I.

To find a number, to which if there be added a half, a third part, and a fourth part of itſelf, the ſum will be 50.

Let it be z : then half of it is $\frac{z}{2}$, a third of it $\frac{z}{3}$ &c.

Therefore, $z+\frac{z}{2}+\frac{z}{3}+\frac{z}{4}=50.$

$$24z+12z+8z+6z=1200$$

$$50z=1200$$

$$z=24.$$

If the operation be more complicated, it may be uſeful to regiſter the ſeveral ſteps of it, as in the following

EXAMPLE II.

A trader allows L. 100 *per annum for the expences of his family, and augments yearly that part of his ſtock which is not ſo expended, by a third part of it; at the end of three years, his original ſtock was doubled: What had he at firſt?*

Let his firſt ſtock be	1	z
Of which he ſpends the firſt year 100 l. and there remains	2	$z-100$
This remainder is increaſed by a third of itſelf	3	$z-100+\frac{z-100}{3}=\frac{4z-400}{3}$
The ſecond year he ſpends 100 l. and there remains	4	$\frac{4z-400}{3}-100=\frac{4z-700}{3}$
He increaſes the remainder by one third of it	5	$\frac{4z-700}{3}+\frac{4z-700}{9}=\frac{16z-2800}{9}$
The third year he ſpends 100 l. and there remains	6	$\frac{16z-2800}{9}-100=\frac{16z-3700}{9}$
He increaſes it by one third	7	$\frac{16z-3700}{9}+\frac{16z-3700}{27}=\frac{64z-1480}{27}$
But at the end of the third year his ſtock is doubled; therefore	8	$\frac{64z-14800}{27}=2z$
By R. 3.	9	$64z-14800=54z$
By R. 1.	10	$10z=14800$
By R. 2.	11	$z=1480$

Therefore his ſtock was L. 1480, which being tried, anſwers the conditions of the queſtion.

CASE II. When there are two unknown quantities.

Rule. *Two independent equations involving the two unknown quantities, muſt be derived from the queſtion. A value of one of the unknown quantities muſt be derived from each of theſe equations: and theſe two values being put equal to each other, a new equation will ariſe, involving only one unknown quantity, and may therefore be reſolved by the preceding rule.*

Two equations muſt be deduced from the queſtion; for, from one including two unknown quantities, it is plain, a known value of either of them cannot be obtained; more than two equations would be unneceſſary, and if any third condition were aſſumed at pleaſure, moſt probably it would be inconſiſtent with the other two, and a queſtion containing three ſuch conditions would be abſurd.

It is to be obſerved, however, that the two conditions, and hence the two equations expreſſing them, muſt be independent, that is, the one muſt not be deducible from the other by algebraical reaſoning; for, otherwiſe, there would in effect be only one equation, under two different forms, from which no ſolution can be derived.

EXAMPLE III.

Two perſons, A *and* B, *were talking of their ages; ſays* A *to* B, *Seven years ago I was juſt three times as old as you were, and ſeven years hence I ſhall be juſt twice as old as you will be: I demand their preſent ages?*

Let the ages of A and B be reſpectively	1	x and y.
Seven years ago they were	2	$x-7$ and $y-7$.
Seven years hence they will be	3	$x+7$ and $y+7$
Therefore by Queſt. 1 and 2.	4	$x-7=3\times\overline{y-7}=3y-21$
Alſo by Queſt. and 3.	5	$x+7=2\times\overline{y+7}=2y+14$
By 4. and tranſp.	6	$x=3y-14$
By 5. and tranſp.	7	$x=2y+7$
By 6. and 7.	8	$3y-14=2y+7$
Tranſp. and 8.	9	$y=21$
By 9. and 6. or 7.	10	$x=49$

The ages of A and B then are 49 and 21, which anſwer the conditions.

The operation might have been a little ſhortened by ſubtracting the 4th from 5th, and thus $14 = -y + 35$; and hence $y = 21$. Therefore, (by 6th) $x = (39 - 14) = 49$.

EXAMPLE IV.

A gentleman diſtributing money among ſome poor people, found he wanted 10 *s. to be able to give* 5 *s. to each; therefore he gives each* 4 *s. only, and finds he has* 5 *s. left.—To find the number of ſhillings and poor people.*

If any queſtion ſuch as this, in which there are two quantities ſought, can be reſolved by means of one letter, the ſolution is in general more ſimple than when two are employed. There muſt be, however, two independent conditions, one of which is uſed in the notation of one of the unknown quantities, and the other gives an equation.

Let

Let the number of poor be	1	z
The number of ſhillings will be	2	$5z-10$
The number of ſhillings is alſo	3	$4z+5$
By 2. and 3.	4	$5z-10=4z+5$
Tranſp.	5	$z=15$

The number of poor therefore is 15, and the number of ſhillings is ($4z+5=$) 65, which anſwer the conditions.

EXAMPLE. V.

A courier ſets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours after, another ſets out from the ſame place, and travels the ſame road at the rate of 5 miles in 3 hours: I demand how long and how far the firſt muſt travel, before he is overtaken by the ſecond?

Let the number of hours which the firſt travelled be	1	y
Then the ſecond travelled	2	$y-8$
The firſt travelled ſeven miles in 5 hours, and therefore in *y* hours	3	$(5:7::y:)\ \frac{7y}{5}$ miles
In like manner the ſecond travelled in *y*— 8 hours	4	$(3:5::y-8:)\ \frac{5y-40}{3}$ miles
But they both travelled the ſame number of miles; therefore by 3. and 4.	5	$\frac{5y-40}{3}=\frac{7y}{5}$
Mult.	6	$25y-200=21y$
Tranſp.	7	$4y=200$
Divid.	8	$y=50$

The firſt then travelled 50 hours, the ſecond (y—8 =) 42 hours.

The miles travelled by each $\left(\frac{7y}{5}=\frac{5y-40}{3}=\right)70.$

Case III. When there are three or more unknown quantities.

Rule. *When there are three unknown quantities, there muſt be three independent equations ariſing from the queſtion; and from each of theſe a value of one of the unknown quantities muſt be obtained. By comparing theſe three values, two equations will ariſe, involving only two unknown quantities, which may therefore be reſolved by the rule for caſe* 2.

In like manner, may the rule be extended to ſuch queſtions as contain four or more unknown quantities; and hence it may be inferred, *That, when juſt as many independent equations may be derived from a queſtion, as there are unknown quantities in it, theſe quantities may be found by the reſolution of equations.*

EXAMPLE VI.

To find three numbers, ſo that the firſt, with half the other two, the ſecond with one third of the other two, and the third with one fourth of the other two, may be equal to 34.

Let the numbers be x, y, z, and the equations are,

	1	$x+\frac{y+z}{2}=34$
	2	$y+\frac{x+z}{3}=34$
	3	$z+\frac{x+y}{4}=34$
From the 1ſt,	4	$x=\frac{68-y-z}{2}$
From the 2d,	5	$x=102-3y-z$
From the 3d,	6	$x=136-4z-y$
From 4th and 5th,	7	$\frac{68-y-z}{2}=102-3y-z$
7th reduced	8	$y=\frac{136-z}{5}$
5=6, and reduced	9	$y=\frac{3z-34}{2}$
8 and 9	10	$\frac{3z-34}{2}=\frac{136-z}{5}$
10th reduced	11	$15z-170=272-2z$
	12	$17z=442$ or $z=26$
By 8 and 5	13	$y=22$ and $x=10$

EXAMPLE VII.

To find a number conſiſting of three places, whoſe digits are in arithmetical proportion; if this number be divided by the ſum of its digits, the quotient will be 48 ; *and if from the number be ſubtracted* 198, *the digits will be inverted.*

Let the 3 digits be	1	x, y, z
Then the number is	2	$100x+10y+z$
If the digits be inverted, it is	3	$100z+10y+x$
The digits are in ar. prop. therefore	4	$x+z=2y$
By queſtion	5	$\frac{100x+10y+z}{x+y+z}=48$
By queſtion	6	$100x+10y+z-198=100z+10y+x$
From 6 and tranſ.	7	$99x=99z+198$
Divid. by 99	8	$x=z+2$
From 4	9	$x=2y-z$
8 and 9	10	$2y-z=z+2$
Tranſp.	11	$y=z+1$
Mult. 5.	12	$100x+10y+z=48x+48y+48z$
Tranſp.	13	$52x=38y+47z$
8 and 11 ſubſtit. for x and y	14	$52z+104=38z+38+47z$
Tranſ.	15	$33z=66$
Divid.	16	$\begin{cases} z=2 \\ y=(z+1=)3 \\ x=(z+2=)4. \end{cases}$

The number then is 432, which ſucceeds upon trial.

It ſometimes happens, that all the unknown quantities, when there are more than two, are not in all the equations expreſſing the conditions, and therefore the preceding rule cannot be literally followed. The ſolution, however, will be obtained by ſuch ſubſtitutions as are uſed in Ex. 7. and 9. or by ſimilar operations, which need not be particularly deſcribed.

Corollary to the preceding rules.

It appears that, in every queſtion, there muſt be as many independent equations as unknown quantities; if there are not, then the queſtion is called *indeterminate*, becauſe it may admit of an infinite number of anſwers; ſince the equations wanting may be aſſumed at pleaſure. There may be other circumſtances, however, to limit the anſwers to one, or a preciſe number, and which, at the ſame time, cannot be directly expreſſed by equations. Such are theſe; that the numbers muſt be integers, ſquares, cubes, and many others. The ſolution of ſuch problems, which are alſo called *Diophantine*, ſhall be conſidered afterwards.

SCHOLIUM.

On many occaſions, by particular contrivances, the operations by the preceding rules may be much abridged. This, however, muſt be left to the ſkill and practice of the learner. A few examples are the following.

1. It is often eaſy to employ fewer letters than there are unknown quantities, by expreſſing ſome of them from a ſimple relation to others contained in the conditions of the queſtion. Thus, the ſolution becomes more eaſy and elegant. (See Ex. 4. 5.).

2. Sometimes it is convenient to expreſs by letters, not the unknown quantities themſelves, but ſome other quantities connected with them, as their ſum, difference, &c. from which they may be eaſily derived. (See Ex. 1. of chap. 5.).

3. In the operation alſo, circumſtances will ſuggeſt a more eaſy road than that

pointed out by the general rules. Two of the original equations may be added together, or may be ſubtracted; ſometimes they muſt be previouſly multiplied by ſome quantity, to render ſuch addition or ſubtraction effectual, in exterminating one of the unknown quantities, or otherwiſe promoting the ſolution. Subſtitutions may be made of the values of quantities, in place of quantities themſelves, and various other ſuch contrivances may be uſed, which will render the ſolution much leſs complicated. (See Ex. 3. 7. and 9.).

C H A P. III.

PART II.

General Solution of Problems.

IN the ſolutions of the queſtions in the preceding part, the given quantities (being numbers) diſappear in the laſt concluſion, ſo that no general rules for like caſes can be deduced from them. But, if letters are uſed to denote the known quantities, as well as the unknown, a general ſolution may be obtained, becauſe, during the whole courſe of the operation, they retain their original form. Hence alſo the connection of the quantities will appear in ſuch a manner as to diſcover the neceſſary limitations of the data, when there are any, which is eſſential to the perfect ſolution of a problem. From this method, too, it is eaſy to derive a ſynthetical demonſtration of the ſolution.

When letters, or any other ſuch ſymbols, are employed to expreſs all the quantities, the algebra is ſometimes called *ſpecious* or *literal.*

EXAMPLE VIII.

To find two numbers, of which the ſum and difference are given.

Let s be the given ſum, and d the given difference; alſo, let x and y be the two numbers ſought.

$$x + y = s$$

$$x - y = d$$

$$\text{Whence}\left\}\begin{aligned} x &= s - y \\ x &= d + y \end{aligned}\right.$$

$$d + y = s - y$$

$$2y = s - d$$

$$y = \frac{s-d}{2}$$

$$\text{And } x = \frac{s+d}{2}$$

Thus, let the given ſum be 100, and the difference 24.

Then $x = \left(\frac{s+d}{2} = \frac{124}{2} =\right) 62$ and $y = \left(\frac{s-d}{2} = \frac{76}{2} =\right) 38$.

In the ſame manner may the canon be applied to any other values of s and d. By reverſing the ſteps in the operation, it is eaſy to ſhew, that if $x = \frac{s+d}{2}$, and $y = \frac{s-d}{2}$, the ſum of x and y muſt be s, and their difference d.

EXAMPLE IX.

If A *and* B *together can perform a piece of work in the time* a, A *and* C *together in the time* b, *and* B *and* C *together in the time* c, *in what time will each of them perform it alone?*

Let *A* perform the work in the time x, *B* in y, and *C* in z; then, as the work is the ſame in all caſes, it may be repreſented by unity.

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If A *and* B *together can perform a piece of work in the time* a, A *and* C *together in the time* b, *and* B *and* C *together in the time* c, *in what time will each of them perform it alone?*

Let A perform the work in the time x, B in y, and C in z; then, as the work is the ſame in all caſes, it may be repreſented by unity.

By the question	1	$(x : 1 :: a :)\frac{a}{x} =$ to the work performed by *A* in *a* days
	2	$(y : 1 :: a :)\frac{a}{y} =$ to the work performed by *B* in *a* days
	3	$(x : 1 :: b :)\frac{b}{x} =$ to the work performed by *A* in *b* days
	4	$(z : 1 :: b :)\frac{b}{z} =$ to the work performed by *C* in *b* days
	5	$(y : 1 :: c :)\frac{c}{y} =$ to the work performed by *B* in *c* days
	6	$(z : 1 :: c :)\frac{c}{z} =$ to the work performed by *C* in *c* days
	7	$\frac{a}{x}+\frac{a}{y}=1$ and $ay+ax=xy$
	8	$\frac{b}{x}+\frac{b}{z}=1$ and $bz+bx=xz$
	9	$\frac{c}{y}+\frac{c}{z}=1$ and $cz+cy=zy$
Mult. 7th by $\frac{bc}{xy}$	10	$\frac{abc}{x}+\frac{abc}{y}=bc$
Mult. 8th by $\frac{ac}{xz}$	11	$\frac{abc}{x}+\frac{abc}{z}=ac$
Mult. 9th by $\frac{ab}{zy}$	12	$\frac{abc}{y}+\frac{abc}{z}=ab$
Add 10th, 11th, 12th	13	$\frac{2abc}{x}+\frac{2abc}{y}+\frac{2abc}{z}=bc+ac+ab$
From 13th ſubt. twice 10th	14	$\frac{2abc}{z}=ac+ab-bc$, & $z=\frac{2abc}{ac+ab-bc}$
From 13th ſubt. twice 11th	15	$\frac{2abc}{y}=bc+ab-ac$, & $y=\frac{2abc}{bc+ab-ac}$
From 13th ſubt. twice 12th	16	$\frac{2abc}{x}=bc+ac-ab$, & $x=\frac{2abc}{bc+ac-ab}$

Example

Example in Numbers.

Let $a=8$ days, $b=9$ days, and $c=10$; then $x=14\frac{34}{49}$, $y=17\frac{23}{41}$, and $z=23\frac{7}{31}$. It appears likewiſe that a, b, c, muſt be ſuch, that the product of any two of them muſt be leſs than the ſum of theſe two multiplied by the third. This is neceſſary to give poſitive values of x, y, and z, which alone can take place in this queſtion. Beſides, if x, y, and z, be aſſumed as any known numbers whatever, and if values of a, b, and c be deduced from ſteps 7th, 8th, and 9th, of the preceding operation, it will appear that a, b, and c, will have the property required in the limitation here mentioned.

If a, b, and c were ſuch, that any of the quantities, x, y, or z, became equal to 0, it implies, that one of the agents did nothing in the work. If the values of any of theſe quantities be negative, the only ſuppoſition which could give them any meaning would be, that ſome of the agents, inſtead of promoting the work, either obſtructed it, or undid it to a certain extent.

EXAMPLE X.

In queſtion 5*th, let the firſt courier travel* p *miles in* q hours; *the ſecond* r *miles in* s *hours; let the interval between their ſetting out be* a;

Then, by working as formerly,

$$x=\frac{qra}{qr-ps}.$$

If particular values be inſerted for theſe letters, a particular ſolution will be obtained for that caſe. Let them denote the numbers in Example 5.

$$\text{Then } x=\left(\frac{qra}{qr-ps}=\frac{5\times5\times8}{5\times5-7\times3}=\frac{200}{4}=\right)50.$$

Here it is obvious, that qr muſt be greater than ps, elſe the problem is impoſſible; for then the value of x would either be infinite or negative. This limitation appears alſo from the nature of the queſtion, as the ſecond courier muſt travel at a greater rate than the firſt, in order to overtake him. For the rate of the firſt courier is to the rate of the ſecond as $\frac{p}{q}$ to $\frac{r}{s}$, that is, as ps to qr; and therefore qr muſt be greater than ps.

SCHOLIUM.

Sometimes when there are many known quantities in a general ſolution, it may ſimplify the operation, to expreſs certain combinations of them by new letters, ſtill to be conſidered as known.

CHAP. IV.

Of Involution and Evolution.

IN order to reſolve equations of the higher orders, it is neceſſary to premiſe the rules of INVOLUTION and EVOLUTION.

LEMMA.

The reciprocals of the powers of a quantity may be expreſſed by that quantity, with negative exponents of the ſame denomination. That is, the ſeries a, 1, $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^m}$, &c. may be expreſſed by a^1, a^0, a^{-1}, a^{-2}, a^{-3}, a^{-m}, &c.

For the rule for dividing the powers of the ſame root was to ſubtract the exponents; if then the index of the diviſor be greater than that of the dividend, the index of the quotient muſt be negative.

Thus, $\frac{a^2}{a^3}=a^{2-3}=a^{-1}$. Alſo, $\frac{a^2}{a^3}=\frac{1}{a^1}$.

$\frac{a^m}{a^m}=a^{m-m}=a^0$. Alſo, $\frac{a^m}{a^m}=1$. And ſo on of others.

Cor. 1. Hence any quantity which multiplies either the numerator or denominator of a fraction, may be tranſpoſed from the one to the other, by changing the ſign of its index.

Thus, $\frac{x}{y^2}=xy^{-2}$. And $\frac{a^2x}{y^3}=\frac{a^2}{y^3x^{-1}}$, &c.

Cor. 2. From this notation, it is evident that theſe *negative powers*, as they are called, are multiplied by adding, and divided by ſubtracting their exponents.

Thus, $a^{-2}\times a^{-3}=a^{-5}$.

Or, $\frac{1}{a^2}\times\frac{1}{a^3}=\frac{1}{a^5}=a^{-5}$.

$\frac{a^{-1}}{a^{-3}}=a^2$; Or, $\frac{1}{a^3}\Big)\frac{1}{a}\Big(\frac{a^3}{a}=a^2$.

I. *Of Involution.*

To find any power of any quantity, is the buſineſs of involution.

Caſe I. When the quantity is ſimple.

Rule. *Multiply the exponents of the letters by the index of the power required, and raiſe the coefficient to the ſame power.*

Thus, the 2d power of a is $a^{1\times 2}=a^2$.

The 3d power of $2a^2$ is $8a^{2\times 3}=8a^6$.

The 3d power of $3ab^3$ is $27a^{1\times 3}b^{3\times 3}=27a^3b^9$.

For the multiplication would be performed by the continual addition of the exponents; and this multiplication of them is equivalent. The ſame rule holds alſo when the ſigns of the exponents are negative.

Rule for the Signs.

If the ſign of the given quantity is +, all its powers muſt be poſitive. If the ſign is —, then all its powers whoſe exponents are even numbers, are poſitive; and all its powers whoſe exponents are odd numbers, are negative.

This is obvious from the rule for the ſigns in multiplication.

The laſt part of it implies the moſt extenſive uſe of the ſigns + and —, by ſuppoſing that a negative quantity may exiſt by itſelf.

Caſe II. When the quantity is compound.

Rule. *The powers muſt be found by a continual multiplication of it by itſelf.*

Thus, the ſquare of $x+\frac{a}{2}$ is found by multiplying it into itſelf. The product is $x^2+ax+\frac{a^2}{4}$. The cube of $x+\frac{a}{2}$ is got by multiplying the ſquare already found by the root, &c.

Fractions are raiſed to any power, by raiſing both numerator and denominator to that power, as is evident from the rule for multiplying fractions, in chap. 1. p. 2.

The involution of compound quantities is rendered much eaſier by the binomial theorem; for which, ſee Chap. VI.

Note. The ſquare of a binomial conſiſts of the ſquares of the two parts, and twice the product of the two parts.

II. *Of Evolution.*

Evolution is the reverſe of involution, and by it powers are reſolved into their roots.

Def. The root of any quantity is expreſſed by placing before it $\sqrt{}$ (called a *radical ſign*) with a ſmall figure above it, denoting the denomination of that root.

Thus, the ſquare root of a is $\sqrt[2]{a}$ or $\sqrt{a}$.

The cube root of bc is $\sqrt[3]{bc}$.

The 4th root of a^2b-x^3 is $\sqrt[4]{a^2b-x^3}$.

The mth root of c^2-dx is $\sqrt[m]{c^2-dx}$.

General Rule for the Signs.

1. *The root of any poſitive power may be either poſitive or negative, if it is denominated by an even number; if the root is denominated by an odd number, it is poſitive only.*

2. *If the power is negative, the root alſo is negative, when it is denominated by an odd number.*

3. *If the power is negative, and the denomination of the root even, then no root can be aſſigned.*

This rule is eaſily deduced from that given in involution, and ſuppoſes the ſame extenſive uſe of the ſigns + and —. If it is applied to abſtract quantities in which a contrariety cannot be ſuppoſed, any root of a poſitive quantity muſt be poſitive only, and any root of a negative quantity, like itſelf, is unintelligible.

In the laſt caſe, though no root can be aſſigned, yet ſometimes it is convenient to ſet the radical ſign before the negative quantity, and then it is called an *impoſſible* or *imaginary root.*

The root of a poſitive power, denominated by an even number, has often the the ſign ± before it, denoting that it may have either + or —.

The radical ſign may be employed to expreſs any root of any quantity whatever; but ſometimes the root may be accurately found by the following rules, and when it cannot, it may often be more conveniently expreſſed by the methods now to be explained.

Caſe I. When the quantity is ſimple.

Rule. *Divide the exponents of the letters by the index of the root required, and prefix the root of the numeral coefficient.*

1. The exponents of the letters may be multiples of the index of the root, and the root of the coefficient may be extracted.

Thus, the ſquare root of $a^4 = a^{\frac{4}{2}} = \pm a^2$

$$\sqrt[3]{27a^6} = 3a^{\frac{6}{3}} = 3a^2$$

$$\sqrt[4]{a^4b^{12}} = a^{\frac{4}{4}}b^{\frac{12}{4}} = \pm ab^3.$$

2. The exponents of the letters may not be multiples of the index of the root, and then they become fractions; and when the root of the coefficient cannot be extracted, it may alſo be expreſſed by a fractional exponent, its original index being underſtood to be 1.

Thus, $\sqrt{16a^3b^2} = 4a^{\frac{3}{2}}b$

$$\sqrt[3]{7ax^3} = 7^{\frac{1}{3}}a^{\frac{1}{3}}x = \sqrt[3]{7} \times a^{\frac{1}{3}}x.$$

As evolution is the reverſe of involution, the reaſon of the rule is evident.

The root of any fraction is found by extracting that root out of both numerator and denominator.

Caſe II. When the quantity is compound.

1. To extract the ſquare root.

RULE.

RULE.

1. *The given quantity is to be ranged according to the powers of the letters, as in diviſion.*

Thus, in the example $a^2+2ab+b^2$, the quantities are ranged in this manner.

2. *The ſquare root is to be extracted out of the firſt term (by preceding rules), which gives the firſt part of the root ſought. Subtract its ſquare from the given quantity, and divide the firſt term of the remainder by double the part already found, and the quotient is the ſecond term of the root.*

Thus, in this example, the remainder is $2ab+b^2$; and $2ab$ being divided by $2a$, the double of the part found, gives $+b$ for the ſecond part of the root.

3. *Add this ſecond part to double of the firſt, and multiply their ſum by the ſecond part: Subtract the product from the laſt remainder,*

der, and if nothing remain, the ſquare root is obtained. But, if there is a remainder, it muſt be divided by the double of the parts already found, and the quotient wlll give the third part of the root; and ſo on.

In the laſt example, it is obvious that $a+b$ is the ſquare root ſought.

The entire operation is as follows :

$$\begin{array}{rl} & a^2+2ab+b^2\,(a+b \\ & a^2 \\ \hline 2a+b\,) & +2ab+b^2 \\ \times b\,) & 2ab+b^2 \\ \hline & \qquad * \end{array}$$

$$\begin{array}{rl} & x^4-ax^2+\frac{a^2}{4}\,(x^2-\frac{a}{2} \\ & x^4 \\ \hline 2x^2-\frac{a}{2}\,\Big) & -ax^2+\frac{a^2}{4} \\ \times-\frac{a}{2}\,\Big) & -ax+\frac{a^2}{4} \end{array}$$

The reaſon of this rule appears from the compoſition of a ſquare.

2. To extract any other root.

Rule. *Range the quantity according to the dimenſions of its letters, and extract the ſaid root out of the firſt term, and that ſhall be the firſt member of the root required. Then raiſe this root to a dimenſion lower by unit, than the number that denominates the root required, and multiply the power that ariſes, by that number itſelf: divide the ſecond term of the given quantity by the product, and the quotient ſhall give the ſecond member of the root required.* In like manner are the other parts to be found, by conſidering thoſe already got as making one term.

Thus, the fifth root of

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5(a+b$

a^5

$5a^4\ )\ 5a^4b$

And $a+b$ raiſed to the 5th power is the given quantity, and therefore it is the root ſought.

In evolution it will often happen, that the operation will not terminate, and the root will be expreſſed by a ſeries.

Thus, the ſquare root of a^2+x^2 becomes a ſeries.

$$\begin{array}{ll}
 & a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}\right., \&c. \\
 & a^2 \\
\hline
2a+\frac{x^2}{2a} & \left.\right) * + x^2 \\
\times\frac{x^2}{2a} & \left.\right) = x^2+\frac{x^4}{4a^2} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3} & \left.\right) * - \frac{x^4}{4a^2} \\
\times-\frac{x^4}{8a^3} & \left.\right) = -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
 & * + \frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
 & \&c.
\end{array}$$

The extraction of roots by ſeries is much facilitated by the binomial theorem (Chap. VI. Sect. 3.). By ſimilar rules, founded on the ſame principles, are the roots of numbers to be extracted.

III. *Of Surds.*

Def. Quantities with fractional exponents are called *Surds*, or *Imperfect Powers.*

Such quantities are alſo called *irrational*, in oppoſition to others with integral exponents, which are called *rational.*

Surds may be expreſſed either by the fractional exponents, or by the radical ſign, the denominator of the fraction being its index; and hence the orders of ſurds are denominated from this index.

In the following operations, however, it is generally convenient to uſe the notation by the fractional exponents.

$a^{\frac{1}{3}} = \sqrt[3]{a}$. $\sqrt{4ab^2} = 2ba^{\frac{1}{2}}$. $\sqrt[4]{a^3b^2} = a^{\frac{3}{4}}b^{\frac{2}{4}}$.

The operations concerning ſurds depend on the following principle. *If the numerator and denominator of a fractional exponent be both multiplied or both divided by the ſame quantity, the value of the power is the ſame.* Thus, $a^{\frac{m}{n}}=a^{\frac{mc}{nc}}$; for, let $a^{\frac{m}{n}}=b$; then $a^m=b^n$, and $a^{mc}=b^{nc}$, and extracting the root nc, $a^{\frac{mc}{nc}}=b^{\frac{nc}{nc}}=b=a^{\frac{m}{n}}$.

Lem. A rational quantity may be put into the form of a ſurd, by reducing its index to the form of a fraction of the ſame value.

Thus, $a=a^{\frac{2}{2}}=\sqrt{a^2}$

$$a^2 b=a^{\frac{6}{3}}b^{\frac{3}{3}}=\sqrt[3]{a^6 b^3}.$$

Prob. I. To reduce ſurds of different denominators to others of the ſame value, and of the ſame denomination.

Rule. *Reduce the fractional exponents to others of the ſame value, and having the ſame common denominator.*

Example. $\sqrt{a}$, $\sqrt[3]{b^2}$, or $a^{\frac{1}{2}}$, $b^{\frac{2}{3}}$; but $a^{\frac{1}{2}}=a^{\frac{3}{6}}$, and $b^{\frac{2}{3}}=b^{\frac{4}{6}}$; therefore $\sqrt{a}$, and $\sqrt[3]{b^2}$ are reſpectively equal to $\sqrt[6]{a^3}$ and $\sqrt[6]{b^4}$.

Prob. II. To multiply and divide ſurds.

1. *When they are ſurds of the ſame rational quantity, add and ſubtract their exponents.*

Thus, $a^{\frac{1}{3}}\times a^{\frac{3}{4}}=a^{\frac{1}{3}+\frac{3}{4}}=a^{\frac{13}{12}}=\sqrt[12]{a^{13}}$.

$$\frac{\sqrt{a^2-b^2}}{\sqrt[3]{a^2-b^2}}=\frac{\left.\overline{a^2-b^2}\right|^{\frac{1}{2}}}{\left.\overline{a^2-b^2}\right|^{\frac{1}{3}}}=\left.\overline{a^2-b^2}\right|^{\frac{1}{6}}=\sqrt[6]{a^2-b^2}.$$

2. *If they are ſurds of different rational quantities, let them be brought to others of the ſame denomination, if already they are not, by prob. 1. Then, by multiplying or dividing theſe rational quantities, their product or quotient may be ſet under the common radical ſign.*

Thus, $\sqrt[m]{a}\times\sqrt[n]{b}=a^{\frac{1}{m}}b^{\frac{1}{n}}=\sqrt[mn]{a^nb^m}$.

$$\frac{\sqrt{a^2-b^2}}{\sqrt{a+b}}=\sqrt{\frac{a^2-b^2}{a+b}}=\sqrt{a-b}.$$

$$\frac{\sqrt[4]{a^3b^2}}{\sqrt[3]{a^2b}}=\frac{a^{\frac{3}{4}}b^{\frac{2}{4}}}{a^{\frac{2}{3}}b^{\frac{1}{3}}}=a^{\frac{3}{4}-\frac{2}{3}}b^{\frac{1}{2}-\frac{1}{3}}=a^{\frac{1}{12}}b^{\frac{1}{6}}=$$
$$a^{\frac{1}{12}}b^{\frac{2}{12}}=\sqrt[12]{ab^2}.$$

If the ſurds have any rational coefficients, their product or quotient muſt be prefixed. Thus, $a\sqrt{n}\times b\sqrt{n}=ab\sqrt{mn}$. It is often convenient, in the operations of this problem, not to bring the ſurds of ſimple quantities to the ſame denomination, but to expreſs their product or quotient without the radical ſign, in the ſame manner as if they were rational quantities. Thus, the product in Ex. 1. may be $a^{\frac{1}{m}}b^{\frac{1}{n}}$, and the quotient in Ex. 3. $a^{\frac{1}{12}}b^{\frac{1}{6}}$.

Cor. If a rational coefficient be prefixed to a radical ſign, it may be reduced to the form of a ſurd by the lemma, and multiplied by this problem; and converſely, if the quantity under the radical ſign be diviſible by a perfect power of the ſame denominomination, it may be taken out, and its root prefixed as a coefficient.

$$a\sqrt{b}=\sqrt{a^2b}\,;\ 2\times\sqrt[3]{a}=\sqrt[3]{8a}.$$

Conv. $\sqrt{a^2b^3}=ab\sqrt{b}\,;\ \sqrt{4a^2-8a^2b}=2a\sqrt{1-2b}.$

Even when the quantity under the radical ſign is not diviſible by a perfect power, it may be uſeful ſometimes to divide ſurds into their component factors, by reverſing the operation of this problem.

Thus, $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$, $\sqrt[3]{a^2t-bx^2}=\sqrt[3]{ba-bx}\times\sqrt[3]{a+x}$
$=b^{\frac{1}{3}}\times\sqrt[3]{a-x}\times\sqrt[3]{a+x}.$

Prob. III. To involve or evolve ſurds.

This is performed by the ſame rules as in other quantities, by multiplying or dividing their exponents by the index of the power or root required.

The notation by negative exponents, mentioned in the lemma at the beginning of this chapter, is applicable to fractional exponents, in the ſame manner as to integers.

SCHO-

SCHOLIUM.

The application of the rules of this chapter to the reſolving of equations, ſhall be explained in the ſucceeding chapters, which treat of the ſolution of the different claſſes of them ; but ſome examples of their uſe in preparing equations for a ſolution, are the following.

If a member of an equation be a ſurd root, then the equation may be freed from any ſurd, by bringing that member firſt to ſtand alone upon one ſide of the equation, and then taking away the radical ſign from it, and raiſing the other ſide to the power denominated by the index of that ſurd.

This operation becomes a neceſſary ſtep towards the ſolution of an equation, when any of the unknown quantities are under the radical ſign.

Example. If $3\sqrt{x^2-a^2}+2y=a+y$

Then $3\sqrt{x^2-a^2}=a-y$

And $9\times\overline{x^2-a^2}=a^2-2ay+y^2$.

If the unknown quantity be found only under the radical ſign, and only of the firſt dimenſion, the equation will become ſimple, and may be reſolved by the preceding rules.

Thus, if $\sqrt[3]{4x+16}+5=9$

Then $\sqrt[3]{4x+16}=4$

And $4x+16=64$

$4x=48$

And $x=12$

If $\sqrt[m]{a^2x-b^2x}=a$

Then $a^2x-b^2x=a^m$

$$x=\frac{a^m}{a^2-b^2}.$$

If the unknown quantity in a final equation has fractional exponents, by means of the preceding rules a new equation may be ſubſtituted, in which the exponents of the unknown quantity are integers.

Thus, if $x^{\frac{1}{2}}+3x^{\frac{2}{3}}=10$, by reducing the ſurds to the ſame denomination, it becomes $x^{\frac{3}{6}}+3x^{\frac{4}{6}}=10$; and if $z=x^{\frac{1}{6}}$, then $z^3+3z^4=10$; and if this equation be reſolved, from a value of z, a value of x may be got by the rules of the next chapter. Thus alſo, if $x+2x^{\frac{1}{2}}-3x^{\frac{1}{3}}=100$. If $x^{\frac{1}{6}}=z$, this equation becomes $z^6+2z^3-3z^2=100$.

In general, if $x^{\frac{p}{q}}+x^{\frac{m}{n}}=a$, by reducing the ſurds to the ſame denomination, $x^{\frac{pn}{nq}}+x^{\frac{qm}{qn}}=a$, and if $x^{\frac{1}{qn}}=x$, then the equation is $z^{pn}+z^{mq}=a$, in which the exponents of z are integers; and z being found, x is to be found from the equation $x^{\frac{1}{qn}}=z$.

C H A P. V.

Equations were divided into orders according to the higheſt index of the unknown quantity in any term. (Chap. III.).

Equations are either *pure* or *adfected.*

Def. 1. A *pure equation* is that in which only one power of the unknown quantity is found.

2. An *adfected equation* is that in which different powers of the unknown quantity are found in the ſeveral terms.

Thus, $a^2+ax^2=b^3$, $ax^2-b^2=m^2+x^2$, are pure equations.

And $x^2-ax=b^2$, $x^3+x^2=17$, are adfected.

I. *Solution of Pure Equations.*

Rule. *Make the power of the unknown quantity to ſtand alone, by the rules formerly given, and then extract the root, of the ſame denomination, out of both ſides, which will give the value of the unknown quantity.*

EXAMPLES.

If $a^2+ax^2=b^3$ $\qquad ax^m-b=x^m-c$

$$ax^2=b^3-a^2 \qquad ax^m-x^m=b-c$$

$$x^2=\frac{b^3-a^2}{a} \qquad x^m=\frac{b-c}{a-1}$$

$$x=\sqrt{\frac{b^3-a^2}{a}} \qquad x=\sqrt[m]{\frac{b-c}{a-1}}.$$

The index of the power may alſo be fractional; as, in the laſt example, m may be any number whatever. Let $m=\frac{1}{2}$, then, as before,

$$x^m = x^{\frac{1}{2}} = \frac{b-c}{a-1}$$

$$\text{And } x = \left.\overline{\frac{b-c}{a-1}}\right|^2 = \frac{b^2-2bc+c^2}{a^2-2a+1}.$$

Sometimes different powers of the unknown quantity are found in the equation, yet the ſeveral terms may form on one ſide a perfect power, of which the root being extracted, the equation will become ſimple.

Thus, if $x^3-12x^2+48x=98$. It is eaſy to obſerve that $x^3-12x^2+48x-64=34$, forming a complete cube, of which the root being extracted, $x-4=\sqrt[3]{34}$. And $x=4+\sqrt[3]{34}$.

EXAMPLE I.

To find four continued proportionals, of which the ſum of the extremes is 56, *and the ſum of the means* 24.

To reſolve the queſtion in general terms, let the ſum of the extremes be a, the ſum of the means b, and let the difference of the extremes be called z, and the difference of the means y. Then, by Ex. 8. Chap. 3.

The

The proportionals are	1	$\frac{a+z}{2} : \frac{b+y}{2} : \frac{b-y}{2} : \frac{a-z}{2}$
Mult. by 2 and ſtill	2	$a+z : b+y : b-y : a-z$
From the three firſt	3	$ab-ay+bz-zy=b^2+2by+y^2$
From the three laſt	4	$ab+ay-bz-zy=b^2-2by+y^2$
3d added to 4th	5	$2ab-2zy=2b^2+2y^2$
4th ſubtr. from 3d	6	$2bz-2ay=4by$
6th reduced	7	$z=\frac{2by+ay}{b}$
7th ſubſt. for x in 5th	8	$2ab-2\times\frac{2by^2+ay^2}{b}=2b^2+2y^2$
Tranſ. and divide 8th by $\frac{2}{b}$	9	$ab^2-b^3=3by^2+ay^2$
	10	$\frac{ab^2-b^3}{3b+a}=y^2$, and $y=\sqrt{\frac{ab^2-b^3}{3b+a}}$
In numbers	11	$y=\sqrt{\frac{ab^2-b^3}{3b+a}}=b\sqrt{\frac{a-b}{3b+a}}=24\sqrt{\frac{32}{128}}=1$
	12	$z=\frac{2b+a}{b}\times y=52.$

Hence the four proportionals are 54, 18, 6, 2; and it appears that B muſt not be greater than *a*, otherwiſe the root becomes impoſſible, and the problem would alſo be impoſſible; which limitation might be deduced alſo from Prop. 25. V. of Euclid.

II.

II. *Solution of adfected Quadratic Equations.*

Adfected equations of different orders are reſolved by different rules, ſucceſſively to be explained.

An adfected quadratic equation (commonly called a quadratic) involves the unknown quantity itſelf, and alſo its ſquare: It may be reſolved by the following

RULE.

1. *Tranſpoſe all the terms involving the unknown quantity to one ſide, and the known terms to the other; and ſo that the term containing the ſquare of the unknown quantity may be poſitive.*

2. *If the ſquare of the unknown quantity is multiplied by any coefficient, all the terms of the equation are to be divided by it, ſo that the coefficient of the ſquare of the known quantity may be* 1.

3. *Add to both ſides the ſquare of half the coefficient of the unknown quantity itſelf, and*

and the ſide of the equation involving the unknown quantity will be a complete ſquare.

4. *Extract the ſquare root from both ſides of the equation, by which it becomes ſimple, and by tranſpoſing the above mentioned half coefficient, a value of the unknown quantity is obtained in known terms, and therefore the equation is reſolved.*

The reaſon of this rule is manifeſt from the compoſition of the ſquare of a binomial, for it conſiſts of the ſquares of the two parts, and twice the product of the two parts. (*Note* at the end of Chap. IV.).

The different forms of quadratic equations, expreſſed in general terms, being reduced by the firſt and ſecond parts of the rule, are theſe:

1. $x^2+ax=b^2$
2. $x^2-ax=b^2$
3. $x^2-ax=-b^2$

Caſe 1. $x^2+ax=b^2$

$$x^2+ax+\frac{a^2}{4}=b^2+\frac{a^2}{4}$$

$$x+\frac{a}{2}=\pm\sqrt{b^2+\frac{a^2}{4}}$$

$$x=\pm\sqrt{b^2+\frac{a^2}{4}}-\frac{a}{2}.$$

Caſe 2. $x^2-ax=b^2$

$$x^2-ax+\frac{a^2}{4}=b^2+\frac{a^2}{4}$$

$$x-\frac{a}{2}=\pm\sqrt{b^2+\frac{a^2}{4}}$$

$$x=\frac{a}{2}\pm\sqrt{b^2+\frac{a^2}{4}}.$$

Caſe 3. $x^2-ax=-b^2$

$$x^2-ax+\frac{a^2}{4}=\frac{a^2}{4}-b^2$$

$$x-\frac{a}{2}=\pm\sqrt{\frac{a^2}{4}-b^2}$$

$$x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-b^2}.$$

Of theſe caſes it may be obſerved,

1. That if it be ſuppoſed, that the ſquare root of a poſitive quantity may be either poſitive or negative, according to the moſt extenſive uſe of the ſigns, every quadratic

equation will have two roots, except ſuch of the third form, whoſe roots become impoſſible.

2. It is obvious that, in the two firſt forms, one of the roots muſt be poſitive, and the other negative.

3. In the third form, if $\frac{a^2}{4}$, or the ſquare of half of the coefficient of the unknown quantity, be greater than b^2, the known quantity, the two roots will be poſitive. If $\frac{a^2}{4}$ be equal to b^2, the two roots then become equal.

But if, in this third caſe, $\frac{a^2}{4}$ is leſs than b^2, the quantity under the radical ſign becomes negative, and the two roots are therefore impoſſible. This may be eaſily ſhewn to ariſe from an impoſſible ſuppoſition in the original equation.

4. If the equation, however, expreſs the relation of magnitudes abſtractly conſidered, where a contrariety cannot be ſuppoſed to take place; the negative roots cannot be of uſe, or rather there are no ſuch roots; for then a negative quantity by itſelf is unintelligible,

intelligible, and therefore the ſquare root of a poſitive quantity muſt be poſitive only. Hence, in the two firſt caſes, there will be only one root; but, in the third, there will be two. For, in this third caſe, $x^2-ax=-b^2$, or $ax-x^2=b^2$, it is obvious that x may be either greater or leſs than $\frac{1}{2}a$, and yet $a-x$ may be poſitive; and hence $\overline{a-x}\times x=ax-x^2$ may alſo be poſitive, and may be equal to a given poſitive quantity b^2: therefore the ſquare root of $x^2-ax+\frac{1}{4}a^2$ may be either $x-\frac{1}{2}a$, or $\frac{1}{2}a-x$, and both theſe quantities alſo poſitive.

Let then $x-\frac{a}{2}=\sqrt{\frac{a^2}{4}-b^2}$, and $x=\frac{a}{2}+\sqrt{\frac{a^2}{4}-b^2}$. Alſo, let $\frac{a}{2}-x=\sqrt{\frac{a^2}{4}-b^2}$, and hence $x=\frac{a}{2}-\sqrt{\frac{a^2}{4}-b^2}$; and theſe are the ſame two poſitive roots as were obtained by the general rule.

The general rule is uſually employed, even in queſtions where negative numbers cannot take place, and then the negative roots of the two firſt forms are neglected.

Sometimes even, only one of the poſitive roots of the third caſe can be uſed, and the other may be excluded by a particular condition in the queſtion. When an impoſſible root ariſes in the ſolution of a queſtion, which has been reſolved in general terms, the neceſſary limitation of the data will be diſcovered.

When a queſtion can be ſo ſtated, as to produce a pure equation, it is generally to be preferred to an adfected. Thus the queſtion in the preceding ſection, by the moſt obvious notation, would produce an adfected equation.

II. *Solution of Queſtions producing Quadratic Equations.*

The expreſſion of the conditions of the queſtion by equations, or the ſtating of it, and the reduction likewiſe of theſe equations, till we arrive at a quadratic equation, involving only one unknown quantity and its ſquare, are effected by the ſame rules which were given for the ſolution of ſimple equations, in Chap. III.

EXAMPLE II.

One lays out a certain ſum of money in goods, which he ſold again for L. 24, and gained as much per cent. as the goods coſt him: I demand what they coſt him?

If the money laid out be	1	y
The gain will be	2	$24-y$
But this gain is ($y:24-y::100$)	3	$\frac{2400-100y}{y}$ per cent.
Therefore by queſtion	4	$y=\frac{2400-100y}{y}$
And by mult. and tr.	5	$y^2+100y=2400$
Completing the ſquare	6	$y^2+100y+\overline{50}\vert^2=2400+2500=4900.$
Extr. the root	7	$y+50=\pm\sqrt{4900}=70$
Tranſp.	8	$y=\pm70-50=20$, or -120.

The anſwer is 20 l. which ſucceeds. The other root, —120, has no place in this example, a negative number being here unintelligible.

Any quadratic equation may be reſolved alſo by the general canons at the beginning of this ſection. That ariſing from this queſtion, (No. 5.), belongs to Caſe I. and $a=100$, $b^2=2400$; therefore,

$$y=\left(-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+b^2}=\right)-\frac{100}{2}\pm$$

$$\sqrt{\frac{100^2}{4}+2400}=20, \text{ or } -120, \text{ as before.}$$

EXAMPLE III.

What two numbers are thoſe, whoſe difference is 15, and half of whoſe product is equal to the cube of the leſſer?

Let the leſſer number be	1	x
The greater is	2	$x+15$
By queſtion	3	$\frac{x^2+15x}{2}=x^3$
Divide by x and mult. by 2.	4	$x+15=2x^2$
4th prepared	5	$x^2-\frac{x}{2}=\frac{15}{2}$
Complete ſquare	6	$x^2-\frac{x}{2}+\frac{1}{16}=\frac{15}{2}+\frac{1}{16}=\frac{121}{16}$
Ext. $\surd$	7	$x-\frac{1}{4}=\pm\frac{11}{4}$
Tranſp.	8	$x=3$, or $-\frac{5}{2}$

The numbers, therefore, are 3 and 18, which anſwer the conditions. This is an example of Caſe 2d, and the negative root is neglected.

A ſolution, indeed, may be repreſented by means of the negative root $-\frac{5}{2}$; for then the other number is $(x+15=)-\frac{5}{2}+15=\frac{25}{2}$. And $\frac{1}{2}\times\frac{25}{2}\times-\frac{5}{2}$, is equal to the cube of $-\frac{5}{2}$. Such a ſolution, though uſeleſs, and even abſurd, it is plain muſt correſpond to the conditions, if thoſe rules with regard to the ſigns be uſed in the application of it, by which it was itſelf deduced. The ſame obſervation may be extended even to impoſſible roots, which being aſſumed as the anſwer of a queſtion, muſt, by reverſing the ſteps of the inveſtigation, correſpond to the original equations, by which the conditions of that queſtion were expreſſed.

EXAMPLE IV.

To find two numbers whoſe ſum is 100, *and whoſe product is* 2059.

Let the given ſum $100=a$, the product $2059=b$, and let one of the numbers ſought be x, the other will be $a-x$. Their product is $ax-x^2$.

Therefore by queſtion	1	$ax-x^2=b$, or $x^2-ax=-b$
Complete the ſquare	2	$x^2-ax+\frac{a^2}{4}=\frac{a^2}{4}-b$
Extr. $\sqrt{}$	3	$x-\frac{a}{2}=\pm\sqrt{\frac{a^2}{4}-b}$
Tranſp.	4	$x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-b}$
And the other number	5	$a-x=\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-b}$.

By inſerting numbers, $x=71$ or 29, and $a-x=29$ or 71, ſo that the two numbers ſought are 71 and 29.

Here it is to be obſerved, that b muſt not be greater than $\frac{a^2}{4}$, elſe the roots of the equation would be impoſſible; that is, the given product muſt not be greater than the ſquare of half the given ſum of the numbers ſought. This limitation can eaſily be ſhewn from other principles; for, the greateſt poſſible product of two parts, into which any number may be divided, is when each of them is a half of it. If b be equal to $\frac{a^2}{4}$, there is only one ſolution, and $x=\frac{a}{2}$, alſo $a-x=\frac{a}{2}$.

EXAMPLE V.

There are three numbers in continual geometrical proportion: The ſum of the firſt and ſecond is 10, *and the difference of the ſecond and third is* 24. *What are the numbers?*

Let the firſt be	1	z
The ſecond will be	2	$10-z$
And the third	3	$34-z$
Since $z:10-z:34-z$	4	$z^2-20z+100=34z-z^2$
Tranſp.	5	$2z^2-54z=-100$
Divid.	6	$z^2-27z=-50$
Compl. the ſquare	7	$z^2-27z+\overline{\frac{27}{2}}\Big\vert^2=\frac{279}{4}-50=\frac{529}{4}$.
Extract the $\sqrt{}$	8	$z-\frac{27}{2}=\pm\sqrt{\frac{529}{4}}=\pm\frac{23}{2}$
Tranſp.	9	$z=\frac{27}{2}\pm\frac{23}{2}=25$, or 2.

But, though there are two poſitive roots in this equation, yet only one of them can here be of uſe, the other being excluded by a condition in the queſtion. For, as the ſum of the firſt and ſecond is 10, 25 cannot be one of them: 2 therefore is the firſt, and the proportionals will be 2, 8, 32.

This reſtriction will alſo appear from the explanation given of the third form, to which this equation belongs. For z may be leſs than $\frac{27}{2}$, but, from the firſt condition of the queſtion, it cannot be greater; hence the quantity $z^2-27z+\overline{\frac{27}{2}}\Big|^2$ can have only one ſquare root, viz. $\frac{27}{2}-z$; and this being put equal to $\sqrt{\frac{529}{4}}$, we have by tranſpoſition $z=\frac{27}{2}-\frac{23}{2}=2$, which gives the only juſt ſolution of the queſtion.

From the other root, indeed, a ſolution of the queſtion may be repreſented by means of a negative quantity. If the firſt then be 25, the three proportionals will be 25, —15, 9. Theſe alſo muſt anſwer the conditions, according to the rules given for negative quantities, though ſuch a ſolution has no proper meaning.

Beſides, it is to be obſerved, that, if the following queſtion be propoſed, '*to find* '*three numbers in geometrical proportion, ſo* '*that the difference of the 1ſt and 2d may* '*be* 10, *and the ſum of the 2d and 3d may* '*be* 24*?*' The equation in ſtep 6th will be produced: For, if the 1ſt be z, the 2d is $z-10$, and the 3d $34-z$, and therefore $34z-z^2=z^2-20z+100$, the very ſame equation as in ſtep 4th. In this queſtion, it is plain that the root 25 only can be uſeful, and the three proportionals are 25, 15, 9.

But the neceſſary limitations of ſuch a problem are properly to be derived from a general notation. Let the ſum of the two firſt proportionals be a, and the difference of the two laſt b. If a is not greater than b, the firſt term muſt be the leaſt; but, if a be greater than b, the firſt term may be either the greateſt or the leaſt.

When the firſt term is the leaſt, the proper notation of the three terms is z, $a-z$,

$a+b-z$, and the equation, when ordered, is $z^2-\frac{3a+b}{2}z=-\frac{a^2}{2}$. If the firſt term be greateſt, and then a is greater than b, the notation of the terms is z, $a-z$, $a-b-z$, and the correſponding equation is $z^2-\frac{3a-b}{2}z=\frac{a^2}{2}$.

Of the firſt of theſe equations it may be obſerved, that, whatever be the value of a and b, the ſquare of $\frac{3a+b}{4}$, viz. of half the coefficient of z, is greater than $\frac{a^2}{2}$, and therefore the roots are always poſſible. If the ſquare be completed, and the roots extracted, they become $z-\frac{3a+b}{4}=\frac{\sqrt{\overline{3a+b}|^2-8a^2}}{4}$, and $\frac{3a+b}{4}-z=\sqrt{\overline{3a+b}|^2-8a^2}$. But in this caſe z is the leaſt of the three terms, and therefore a is greater than $2z$, or $\frac{a}{2}$ is greater than z; much more then is $\frac{3a+b}{4}$ greater than z; and therefore the ſecond root only can be admitted, and $z=\frac{3a+b-\sqrt{\overline{3a+b}|^2-8a^2}}{4}$ is the only proper ſolution.

In the ſecond equation, ſince a is greater than b, $\frac{3a-b}{2}$ muſt be always poſitive, and therefore the equation is neceſſarily of the third form. But the roots are poſſible only when $\overline{\frac{3a-b}{4}}\Big|^2$ is not leſs than $\frac{a^2}{2}$, that is, when a^2+b^2 is not leſs than $6ab$, or when $a-b$ is not leſs than $2\sqrt{ab}$. When the roots are poſſible, z may be either greater or leſs than $\frac{3a-b}{4}$, and hence each root gives a proper ſolution; therefore, $z=\frac{3a-b\pm\sqrt{\overline{3a-b}|^2-8a^2}}{4}$.

Ex. Let $a=40$, and $b=6$. The firſt term in this caſe may be aſſumed either as the greateſt or the leaſt. And, firſt, if z be the greateſt, the roots of the equation will be poſſible, ſince $(a^2+b^2=)$ 1636 is greater than $(6ab=)$ 1440. The two values of z are 32 and 25, and the proportionals are either 32, 8, 2, or 25, 15, 9. 2*dly*, If z

be aſſumed the leaſt of the proportionals, the two roots of the equation are poſſible, but one of them only can be applied, which is 17.635 nearly, and the three proportionals are 17.635, 22.365, and 28.365, nearly, the roots of the equation being incommenſurate.

In like manner, may the limitations of the other queſtion above mentioned be aſcertained.

Though the preceding queſtions have been ſo contrived that the anſwers may be integers, yet in practice it will moſt commonly happen that they muſt be ſurds. When in any queſtion the root of a number which is not a perfect ſquare is to be extracted, it may be continued in decimals, by the common arithmetical rule, to any degree of accuracy which the nature of the ſubject may require.

SCHOLIUM.

An equation, in the terms of which two powers only of the unknown quantity are found, and ſuch that the index of the one is double that of the other, may, by the preceding rules, be reduced to a pure equation, and therefore may be reſolved by Sect. I. of this Chapter.

Such an equation may generally be repreſented thus,

$$x^{2m} \pm ax^m = \pm b^n$$

Let $x^m = z$, then $z^2 \pm az = \pm b^n$

$$\text{And } x^m (= z) = \pm \frac{a}{2} \pm \sqrt{\frac{a^2}{4} \pm b^n}$$

$$\text{Therefore } x = \sqrt[m]{\pm \frac{a}{2} \pm \sqrt{\frac{a^2}{4} \pm b^n}}$$

EXAMPLE XV.

To find two numbers, of which the product is 100, *and the difference of their ſquare root* 3.

Let

Let the leſs be x, the greater is	1	$\frac{100}{x}$
By queſtion	2	$\frac{10}{\sqrt{x}}-\sqrt{x}=3$
	3	$10-x=3\sqrt{x}=3x^{\frac{1}{2}}$
	4	$x+3x^{\frac{1}{2}}=10$
	5	$x+3x^{\frac{1}{2}}+\frac{9}{4}=10+\frac{9}{4}=\frac{49}{4}$
	6	$x^{\frac{1}{2}}+\frac{3}{2}=\pm\frac{7}{2}$ and $x^{\frac{1}{2}}=2$ or -5
	7	$x=4$ or $x=25$.

If $x=4$ the other number is 25, and this is the proper ſolution, for x was ſuppoſed to be the leaſt. In this caſe, indeed, the negative root of the equation being applied, according to the rules for negative quantities, gives a poſitive anſwer to the queſtion, and if $x=25$, the other number is 4.

The ſame would have been got, by ſubſtituting in the general theorem $m=\frac{1}{2}$, $a=3$, and $b^n=10$; or, if the leſs number had been called x^2, the equation would not have had fractional exponents.

C H A P.

C H A P. VI.

Of Indeterminate Problems.

IT was formerly obſerved, (Chap. III.) that if there are more unknown quantities in a queſtion, than equations by which their relations are expreſſed, it is indetermined ; or it may admit of an infinite number of anſwers. Other circumſtances, however, may limit the number in a certain manner ; and theſe are various, according to the nature of the problem. The contrivances by which ſuch problems are reſolved are ſo very different in different caſes, that they cannot be comprehended in general rules.

EXAMPLE I.

To divide a given ſquare number into two parts, each of which ſhall be a ſquare number.

There are two quantities ſought in this queſtion, and there is only one equation expreſſing their relation; but it is required alſo, that they may be rational, which circumſtance cannot be expreſſed by an equation; another condition therefore muſt be aſſumed in ſuch a manner as to obtain a ſolution in rational numbers.

Let the given ſquare be a^2, let one of the ſquares ſought be x^2, the other is a^2-x^2. Let $rx-a$ alſo be a ſide of the laſt ſquare; therefore $r^2x^2-2rxa+a^2=a^2-x^2$

By tranſp. $r^2x^2+x^2=2rxa$

Divide by x $r^2x+x=2ra$

Therefore $x=\frac{2ra}{r^2+1}$

And $rx-a=\frac{2r^2a}{r^2+1}-a=\frac{r^2-1}{r^2+1}a.$

Let r, therefore, be aſſumed at pleaſure, and $\frac{2ra}{r^2+1}$, $\frac{r^2-1}{r^2+1}a$, which muſt always be rational, will be the ſides of the two ſquares required.

Thus, if $a^2=100$. Then, if $r=3$, the ſides of the two ſquares are 6 and 8, for $36+64=100$.

Alſo, let $a^2=64$. Then, if $r=2$, the ſides of the ſquares are $\frac{32}{5}$ and $\frac{24}{5}$; and $\frac{1024}{25}+\frac{576}{25}=\frac{1600}{25}=64$.

The reaſon of the aſſumption of $rx-a$ as a ſide of the ſquare a^2-x^2, is that being ſquared and put equal to this laſt, the equation manifeſtly will be ſimple, and the root of ſuch an equation is always rational.

EXAMPLE II.

To find two ſquare numbers whoſe difference is given.

Let x^2 and y^2 be the ſquare numbers, and a their difference.

Put $\frac{z+v}{2}=x$, and $\frac{z-v}{2}=y$.

$$\frac{z^2+2zv+v^2}{4}=x^2$$

$$\frac{z^2-2zv+v^2}{4}=y^2$$

$$zv=x^2-y^2=a.$$

If x and y are required only to be rational, then take v at pleaſure, and $z=\frac{a}{v}$, whence x and y are known.

But, if x and y are required to be whole numbers, *Take for* z *and* v *any two factors that produce* a, *and are both even or both odd numbers, and this is poſſible only where* a *is either an odd number greater than* 1, *or a number diviſible by* 4. *Then* $\frac{z+v}{2}$ *and* $\frac{z-v}{2}$ *are the numbers ſought.*

For the product of two odd numbers is odd, and that of two even numbers is diviſible by 4. Alſo, if z and v are both odd, or both even, $\frac{z+v}{2}$ and $\frac{z-v}{2}$ muſt be integers.

Ex. 1. If $a=27$ take $v=1$, then $z=27$; and the ſquares are 196 and 169: or z may be 9 and $v=3$, and then the ſquares are 36 and 9.

2. If $a=12$, take $v=2$, and $z=6$; and the ſquares are 16 and 4.

EXAMPLE III.

To find a ſum of money in pounds and ſhillings, whoſe half is juſt its reverſe.

Note. The reverſe of a ſum of money, as 8 l. 12 *s.* is 12 l. 8 s.

Let x be the pounds, and y the ſhillings.

The ſum required is $20x+y$

Its reverſe is - $20y+x$

Therefore $\frac{20x+y}{2}=20y+x$

$$20x+y=40y+2x$$

$$18x=39y$$

$$x:y::(39:18::)\ 13:6.$$

In this equation there are two unknown quantities; and, in general, any two numbers of which the proportion is that of 13 to 6, will agree to it.

But, from the nature of the queſtion, 13 and 6 are the only two that can give the proper anſwer, viz. 13 l. 6 s. for its reverſe 6 l. 13 s. is juſt its half.

The ratio of x and y is expreſſed in the loweſt integral terms by 13 and 6; any other expreſſion of it, as the next greater 26 and 12, will not ſatisfy the problem, as 12 l. 26 s. is not a proper notation of money in pounds and ſhillings.

CHAP.

CHAP. VII.

Demonſtration of Theorems by Algebra.

ALGEBRA may be employed for the demonſtration of Theorems, with regard to all thoſe quantities concerning which it may be uſed as an analyſis, and from the general method of notation and reaſoning, it poſſeſſes the ſame advantages in the one as in the other. The three firſt ſections of this Chapter contain ſome of the moſt ſimple properties of *ſeries* which are of frequent uſe; and the laſt, miſcellaneous examples of the properties of algebraical quantities and numbers.

I. *Of Arithmetical Series.*

Def. When a number of quantities increaſe or decreaſe by the ſame common difference, they form an *Arithmetical Series.*

Thus, a, $a+b$, $a+2b$, $a+3b$, &c. x, $x-b$, $x-2b$, &c.

Alſo, 1, 2, 3, 4, 5, 6, &c.; and 8, 6, 4, 2, &c.

Prop. In an arithmetical ſeries, the ſum of the firſt and laſt terms is equal to the ſum of any two intermediate terms, equally diſtant from the extremes.

Let the firſt term be a, the laſt x, and b the common difference; then $a+b$ will be the ſecond, and $x-b$ the laſt but one, &c.

Thus, a, $a+b$, $a+2b$, $a+3b$, $a+4b$, &c.
x, $x-b$, $x-2b$, $x-3b$, $x-4b$, &c.

It is plain, that the terms in the ſame perpendicular rank are equally diſtant from the extremes, and that the ſum of any two in it is $a+x$, the ſum of the firſt and laſt.

Cor. 1. Hence the ſum of all the terms of an arithmetical ſeries is equal to the ſum of the firſt and laſt, taken half as often as there are terms.

Therefore, if n be the number of terms, and s the ſum of the ſeries; $s=\overline{a+x}\times\frac{n}{2}$. If $a=0$, then $s=\frac{nx}{2}$.

Cor. 2. The ſame notation being underſtood, ſince any term in the ſeries conſiſts of a, the firſt term, together with b taken as often as the number of terms preceding it, it follows that $x=a+\overline{n-1}\times b$, and hence $s=\overline{2a+\overline{n-1}\times b}\times\frac{n}{2}$; or by multiplication, $s=\frac{2an+n^2b-nb}{2}$. Therefore, from the firſt term, the common difference, and number of terms being given, the ſum may be found.

Ex. Required the ſum of 50 terms of the ſeries 2, 4, 6, 8, &c.

$$s=\frac{2\times2\times50+50^2\times2-50\times2}{2}=\frac{5100}{2}=2550.$$

Cor. 3. Of the firſt term, common difference, ſum and number of terms, any three being given, the fourth may be found by reſolving the preceding equation; a, b, s, and n being ſucceſſively conſidered as the unknown quantity. In the three firſt caſes the equation is ſimple, and in the laſt it is quadratic.

II. *Of Geometrical Series.*

Def. When a number of quantities increaſe by the ſame multiplier, or decreaſe by the ſame diviſor, they form a *Geometrical Series*. This common multiplier or diviſor is called the *common ratio.*

Thus a, ar, ar^2, &c. a, $\frac{a}{r}$, $\frac{a}{r^2}$, $\frac{a}{r^3}$, &c. 1, 2, 4, 8, &c.

Prop. I. The product of the extremes in a geometrical ſeries is equal to the product of any two terms, equally diſtant from the extremes.

Let a be the firſt term, y the laſt, r the common ratio; then the ſeries is,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \&c.$$

$$y,\ \frac{y}{r},\ \frac{y}{r^2},\ \frac{y}{r^3},\ \frac{y}{r^4},\ \&c.$$

It is obvious that any term in the upper rank is equally diſtant from the beginning, as that below it from the end; and the product of any two ſuch is equal to ar, the product of the firſt and laſt.

Prop. II. The ſum of a geometrical ſeries wanting the firſt term, is equal to the ſum of all but the laſt term, multiplied by the common ratio.

For, aſſuming the preceding notation of a ſeries, it is plain that

$$ar+ar^2+ar^3,\ \&c.\ \ldots +\frac{y}{r^3}+\frac{y}{r^2}+\frac{y}{r}+y=$$
$$=r\times\overline{a+ar+ar^2,\ \&c.\,.+\frac{y}{r^4}+\frac{y}{r^3}+\frac{y}{r^2}+\frac{y}{r}}.$$

Cor. 1. Therefore s being the ſum of the ſeries,

$$\overline{s-y}\times r=s-a. \text{ And } s=\frac{yr-a}{r-1}$$

Hence s can be found from a, y, and r; and any three of the four being given, the fourth may be found.

Cor. 2. Since the exponent of r in any term is equal to the number of terms preceding it; hence, in the laſt term, its exponent will be $n-1$; the laſt term, therefore $y=ar^{n-1}$, and $s=\frac{r^n-a}{r-1}=a\times\frac{r^n-1}{r-1}$. Hence of theſe four, s, a, r, n, any three being given, the fourth may be found by the ſolution of equations. If n is not a ſmall number, the caſes of this problem will be moſt conveniently ſolved by logarithms; and of ſuch ſolutions there are examples in the Appendix to this part.

Cor. 3. If the ſeries decreaſes, and the number of terms is infinite, then according to this notation, a, the leaſt term, will be 0, and $s=\frac{yr}{r-1}$, a finite ſum.

Ex. Required the ſum of the ſeries 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. to infinity.

Here $y=1$, and $r=2$. Therefore $s=\frac{1\times2}{2-1}=2$.

What are called in arithmetic, repeating and circulating decimals, are truly geometrical decreaſing ſerieſes, and therefore may be ſummed by this rule.

Thus $.333$, &c. $=\frac{3}{10}+\frac{3}{100}+$ &c. is a geometrical ſeries in which $y=\frac{3}{10}$ and $r=10$, therefore $s=\frac{yr}{r-1}+\frac{3\times10}{10\times\overline{10-1}}=\frac{1}{3}$.

Thus alſo $.2424$, &c, $=\frac{8}{33}$, for here $y=\frac{24}{100}$ and $r=100$, therefore $s=\frac{24\times100}{100\times\overline{100-1}}=\frac{24}{99}=\frac{8}{33}$.

III. *Of Infinite Series.*

It was obſerved, (Chap. I. and IV.) that in many caſes, if the diviſion and evolution of compound quantities be actually performed,

formed, the quotients and roots can only be expreſſed by the ſeries of terms, which may be continued *ad infinitum*. By comparing a few of the firſt terms, the law of the progreſſion of ſuch a ſeries will frequently be diſcovered, by which, it may be continued without any further operation. When this cannot be done, the work is much facilitated by ſeveral methods; the chief of which is that by the *binomial theorem*.

Theorem.

Any binomial (as $a+b$) may be raiſed to any power (m) by the following rules.

1. From inſpecting a table of the powers of a binomial obtained by multiplication, it appears that the terms, without their coefficients are a^{m}, $a^{m-1}b$, $a^{m-2}b^{2}$, $a^{m-3}b^{3}$, &c.

2. The coefficients of theſe terms will be found by the following rule.

Divide the exponent of a *in any term by the exponent of* b *increaſed by* 1, *and the quotient multiplied by the coefficient of that term will give the coefficient of the next following term.*

This rule is found, upon trial in the table of powers, to hold univerſally. The coefficient of the firſt term is always 1; and by applying the general rule now propoſed, the coefficients of the terms in order will be as follows, 1, m, $m\times\frac{m-1}{2}$, $m\times\frac{m-1}{2}\times\frac{m-2}{3}$, &c. They may be more conveniently expreſſed thus, 1, Am, B$\times\frac{m-1}{2}$, C$\times\frac{m-2}{3}$, D$\times\frac{m-3}{4}$, &c. the capitals denoting the preceding coefficient. Hence $\overline{a+b}|^m = a^m + \mathrm{A}ma^{m-1}b + \mathrm{B}\times\frac{m-1}{2}\times a^{m-2}b^2 + \mathrm{C}\times\frac{m-2}{3}a^{m-3}b^3$, &c. This is the celebrated *binomial theorem*. It is deduced here by induction only, but it may be rigidly demonſtrated, though upon principles which do not belong to this place.

Cor. 1. As m may denote any number integral or fractional, poſitive or negative; hence the diviſion, involution and evolution of a binomial, may be performed by this theorem.

Ex. 1. Let $m=\frac{1}{2}$, then $\overline{a+b}|^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}b+\frac{1}{2}\times-\frac{1}{4}\times a^{-\frac{3}{2}}b^2+$, &c. This being applied to the extraction of the ſquare root of a^2+x^2 (by inſerting a^2 for a and x^2 for b) the ſame ſeries reſults as formerly. (Chap. IV.).

Ex. 2. If $\frac{1}{1-r}$ is to be turned into an infinite ſeries, ſince $\frac{1}{1-r}=1\times\overline{1-r}|^{-1}$, let $a=1$, $b=-r$, and $m=-1$; and the ſame ſeries will ariſe as was obtained by diviſion. (Chap. I.).

In like manner, $\frac{r^2}{\sqrt{2rz-z^2}}=r^2\times\overline{2rz-z^2}|^{-\frac{1}{2}}$ may be expreſſed by an infinite ſeries, ſuppoſing $a=2rz$, $b=-z^2$, and $m=-\frac{1}{2}$, and then multiplying that ſeries by r^2.

Cor. 2. This theorem is uſeful alſo in diſcovering the *law* of an infinite ſeries, produced by diviſion or evolution. Thus, the ſeries expreſſing the ſquare root of a^2+x^2, conſiſts of a, together with a ſeries of fractions, in the numerators of which are the even powers of x, and in the denominators the odd powers of a. The numeral coefficients of the terms of the whole ſeries, as deduced by the theorem, will be:

$$1,\ +\frac{1}{2\times 1},\ -\frac{1\times 1}{2.2\times 1.2},\ +\frac{1\times 1.3}{2.2.2\times 1.2.3},\ -\frac{1\times 1.3.5}{2.2.2.2\times 1.2.3.4},$$

&c. the point being uſed (as it often is) to expreſs the product of the numbers between which it is placed. The law of continuation is obvious, and the ſeries may be carried on to any number of terms, without uſing the theorem. Hence alſo the coefficient of the nth term is $\frac{1\times 1.3.5\ \&c.\ ..\ (n-2\ \text{terms})}{2^{n-1}\times 1.2.3.4\ \&c,\ (n-1)}$; and it is $+$, if n is an even number, and $-$, if n is odd.

Note, If the binomial is $a+b$, the ſigns of the terms of any power are all poſitive: If it is $a-b$, the alternate terms are negative, beginning at the ſecond. This theorem may be applied to quantities which conſiſt of more than two parts, by ſuppoſing them diſtinguiſhed into two, and then ſubſtituting for the powers of theſe compound parts, their values, to be obtained alſo, if required, from the theorem, Thus, $\overline{a+b}\,\overline{+c}|^2 = \overline{\overline{a+b}+c}|^2$.

SCHOLIUM.

An infinite ſeries may itſelf be multiplied or divided by another; it may be involved or evolved, and various other operations may be performed upon it which are neceſſary in the higher parts of algebra. The methods for finding the ſum depend upon other principles.

IV. *Properties of Numbers.*

Theor. I. The ſum of two quantities multiplied by their difference is equal to the difference of their ſquares.

Let the quantities be repreſented by a and b, then $\overline{a+b} \times \overline{a-b} = a^2 - b^2$, as appears by performing the operation.

Cor. If a and b be any two quantities of which the ſum may be denoted by s, the difference by d, and their product by p, then the following propoſitions will be true.

1. $a^2+b^2=s^2-2p$. 2. $a^2-b^2=sd$.
3. $a^3+b^3=s^3-3ps$. 4. $a^3-b^3=s^2d-dp$.
5. $a^4+b^4=s^4-4ps^2+2p^2$. 6. $a^4-b^4=s^3d-2sdp$, &c.

It is unneceſſary to expreſs theſe propoſitions in words, and the demonſtrations are very eaſy, by raiſing $a+b$ to certain powers, and making proper ſubſtitutions.

Theor. II. The ſum of any number of terms (n) of the odd numbers 1, 3, 5, &c. beginning with 1, is equal to the ſquare of that number (n).

In the rule for ſumming an arithmetical ſeries, let $a=1$, $b=2$, and $n=n$, and the ſum of this ſeries will be $s = \frac{2an + n^2b - nb}{2} = \frac{2n^2}{2} = n^2$ Q. E. D.

THEOR. III. The difference of any two ſquare numbers is equal to the ſum of the two roots, together with twice the ſum of the numbers in the natural ſcale between the two roots.

Let the one number be p, and the other $p+n$, the intermediate numbers are $p+1$, $p+2$, . . . &c. $p+n-1$. The difference of the ſquares of the given numbers is $2pn+n^2$; the ſum of the two roots is $2p+n$, and twice the ſum of the ſeries $\overline{p+1}+\overline{p+2}$ . . . &c. $\overline{p+n-1}$ is (by Cor. 1. 1ſt ſect. of this chap.) $2s = \overline{2p+n} \times \overline{n-1}$, viz. the ſum of the firſt and laſt multiplied by the number of terms, and it is plain that $2p+n+\overline{2p+n} \times \overline{n-1} = 2pn+n^2$. Therefore, &c.

LEM 1. Let r be any number, and n any integer, r^n-1 is diviſible by $r-1$.

The quotient will be $r^{n-1}+r^{n-2}$, &c. till the index of r be 0, and then the laſt term of it will be 1; for, if this ſeries be multiplied by the diviſor $r-1$, it will produce the dividend r^n-1. It will appear alſo by performing the diviſion, and inſerting for n any number.

LEM. 2. Let r be any number, and n any integer odd number, r^n+1 is diviſible by $r+1$. Alſo, if n is any even number, r^n-1 is diviſible by $r+1$.

The quotient in both caſes is $r^{n-1}-r^{n-2}+r^{n-3}$, &c. till the exponent of r be 0, and the laſt term $r^0=1$. If this ſeries conſiſt of an odd number of terms, and be multiplied by $r+1$ the diviſor, the product is r^n+1 the dividend. If the ſeries conſiſt of an even number of terms, the product is r^n-1; but it is plain that the number of terms will be odd only when n is odd, and even only when n is even. The concluſion will be manifeſt by performing the diviſion.

LEM. 3. If r is the root of an arithmetical ſcale, any number in that ſcale may be repreſented in the following manner, a, b, c,

&c. being the coefficients or digits, $a+br+cr^2+dr^3+er^4$, &c.

THEOR. IV. If from any number in the general ſcale now deſcribed, the ſum of its digits be ſubtracted, the remainder is diviſible by $r-1$.

The number is $a+br+cr^2+dr^3$, &c. and the ſum of the digits is $a+b+c+d$, &c. Subtracting the latter from the former, the remainder is $br-b+cr^2-c+dr^3-d$, &c. $=b\times\overline{r-1}+c\times\overline{r^2-1}+d\times\overline{r^3-1}$ &c. But (by Lem. 1.) r^n-1 is diviſible by $r-1$, whatever integer number n may be, and therefore any multiple of r^n-1 is alſo diviſible by $r-1$. Hence each of the terms, $b\times\overline{r-1}$, $c\times\overline{r^2-1}$, &c. is diviſible by $r-1$: and therefore the whole is diviſible by $r-1$.

COR. I. Any number, the ſum of whoſe digits is diviſible by $r-1$, is itſelf diviſible by $r-1$. Let the number be called N, and the ſum of the digits D; then, by this Prop. $N-D$ is diviſible by $r-1$, and D is ſuppoſed to be diviſible by $r-1$; therefore it is plain that N muſt alſo be diviſible by $r-1$.

COR. 2. Any number, the ſum of whoſe digits is diviſible by an aliquot part of $r-1$, is alſo diviſible by that aliquot part. For, let N and D denote as before; and ſince $N-D$ (Theor. 4.) is diviſible by $r-1$, it is alſo diviſible by an aliquot part of $r-1$; but D is diviſible by an aliquot part of $r-1$, therefore N is alſo diviſible by that aliquot part.

COR. 3. This theorem, with the corollaries, relates to any ſcale whatever. It includes therefore the well known property of 9, and of 3 its aliquot part, in the decimal ſcale; for, ſince $r=10$, $r-1=9$.

THEOR. V. In any number, if from the ſum of the coefficients of the odd powers of r, the ſum of the coefficients of the even powers be ſubtracted, and the remainder added to the number itſelf, the ſum will be diviſible by $r+1$.

In the number $a+br+cr^2+dr^3+er^4+fr^5$, &c. the ſum of the coefficients of the odd powers of r is $b+d+f$, &c. the ſum of the coefficients of the even powers of r is $a+c+e$, &c. If the latter ſum be ſubtracted from the former, and the remainder added to the given number, it makes $br+b+cr^2-c+dr^3+d+er^4-e+fr^5+f$, &c. $=b\times\overline{r+1}+c\times\overline{r^2-1}+d\times\overline{r^3+1}+e\times\overline{r^4-1}+f\times\overline{r^5+1}$, &c. But (by Lem. 2.) $r+1$, r^2-1, r^3+1, &c. are each diviſible by $r+1$, and therefore any multiples of them are alſo diviſible by $r+1$, hence the whole number is diviſible by $r+1$.

Cor. 1. If the difference of the ſum of the even digits, and the ſum of the odd digits of any number be diviſible by $r+1$, the number itſelf is diviſible by $r+1$.

Let the ſum of the even digits (that is, the coefficients of the odd powers of r) be D, the ſum of the odd digits be d, and let the number be N. Then, by the theorem, $N+D-d$ is diviſible by $r+1$, and it is ſuppoſed that $D-d$ is diviſible by $r+1$; therefore N is diviſible by $r+1$.

Cor. 2. In like manner, if $D-d$ is diviſible by an aliquot part of $r+1$, N will be diviſible by that aliquot part.

Cor. 3. If a number want all the odd powers of r, or if it want all the even powers of r, and if the ſum of its digits be diviſible by $r+1$, that number is diviſible by $r+1$.

Cor. 4. In the common ſcale $r+1=11$, which therefore will have the properties mentioned in this theorem, and the corollaries. Thus, in the number 64834, the ſum of the even digits is 7, the ſum of the odd digits is 18, and the difference is 11, a number diviſible by 11, the given number therefore (Cor. 1.) is diviſible by 11. Thus alſo, the ſum of the digits of 7040308 is diviſible by 11, and therefore the number is diviſible by 11. (Cor. 3).

SCHOLIUM.

Theſe theorems relate to any ſcale whatever, and therefore the properties of $r-1$

in Theor. 4. would in a ſcale of *eight* belong to *ſeven*, and thoſe in Theor. 5. to *nine*. If *twelve* was the root of the ſcale, the former properties would belong to *eleven*, and the latter to *thirteen*,

APPENDIX

TO

PART I.

ALGEBRA may be employed in expreſſing the relations of magnitude in general, and in reaſoning with regard to them. It may be uſed in deducing not only the relations of number, but alſo thoſe of extenſion, and hence thoſe of every ſpecies of quantity expreſſible by numbers or extended magnitudes. In this Appendix are mentioned ſome examples of its application to other parts of mathematics, to phyſics, and to the practical calculations of buſineſs. The principles and ſuppoſitions peculiar to theſe ſubjects, which are neceſſary in directing both the algebraical operations, and the concluſions to be drawn from them, are here aſſumed as juſt and proper.

I.

Algebra has been ſucceſsfully applied to almoſt every branch of mathematics; and the principles of theſe branches are often advantageouſly introduced into algebraical calculations.

The application of it to *geometry* has been the ſource of great improvements in both theſe ſciences. On account of its extent and importance, it is here omitted, and the principles of it are more particularly explained in the third part of theſe elements.

In this place ſhall be given an example of the uſe of logarithms in reſolving algebraical queſtions.

Note. When logarithms are uſed, let (*l.*) denote the logarithm of any quantity before which it is placed.

Ex. *To find the number of terms of a geometrical ſeries, of which the ſum is* 511, *the firſt term* 1, *and the common ratio* 2.

From Sect. 2. Chap. 6. it appears that $s=\frac{ar^n-a}{r-1}$, and in this problem, s, r, and a

are given, and n is to be found. By reducing the equation, $r^n=\frac{s\times\overline{r-1}+a}{a}$, and from the known property of logarithms $n\times l.r= l.\overline{s\times\overline{r-1}+a}-l.a$, and $n=\frac{l.\overline{s\times\overline{r-1}+a}-l.a}{l.r}$. But here $s=511$, $a=1$, $r=2$, and $n=\frac{l.512}{l.2}=\frac{2.7092700}{0.3010300}=9$.

In like manner, may any ſuch equation be reſolved, when the only unknown quantity is an exponent, and when it is the exponent only of one quantity.

Ex. 2. An equation of the following quadratic form $a^{2x}\pm 2ba^x=\pm c$ may be reſolved by logarithms. 1ſt, By ſcholium of chap. 5. $a^x=\mp b\pm\sqrt{b^2\pm c}$. And then x is diſcovered in the ſame manner as in the preceding example. Thus, let $a=2$, $b=10$, and $c=96$, and the equation $2^{2x}-20\times 2^x=-96$. 1ſt, $2^x=10\pm\sqrt{4}=12$ or 8. If $2^x=8$, then $x=\frac{l.8.}{l.2.}=3$, and $2^6-20\times 2^3=-96$ is a true equation. If $2^x=12$, then $x=\frac{l.12.}{l.2}=\frac{1.0791812}{0.3010300}=3.5849$, and this number being inſerted for x in the given equation, by means of logarithms, will anſwer the conditions.

Ex. 3. The ſum of 2000l. has been out at intereſt for a certain time, and 500 l. has been at intereſt double of that time, the whole arrear now due reckoning 4 *per cent.* compound intereſt, is 6000 l. What were the times?

By the rules in the third part of this appendix for compound intereſt, it is plain that if $R=1{,}04$, and the time at which the 2000 l. is at intereſt be x, the arrear of it will be $2000\times R^{x}$. The arrear of the 500 l. is $500\times R^{2x}$, hence $500\times R^{2x}+2000\times R^{x}=6000$. This reſolved gives $R^{x}=2$, and $x=\frac{l.2.}{l.R}=17.67+$, nearly, that is, 17 years and 8 months nearly, and the double is 35 years and 4 months; which anſwer the conditions.

II. *Application of Algebra to Phyſics.*

Phyſical quantities which can be divided into parts, that have proportions to each other, the ſame as the proportions of lines to lines, or of numbers to numbers, may be expreſſed by lines and numbers, and therefore by algebraical quantities. Hence theſe mathematical notations may be conſidered as the meaſures of ſuch phyſical quantities; they may be reaſoned upon according to the principles of algebra, and from ſuch reaſonings, new relations of the quantities which they repreſent, may be diſcovered.

In thoſe branches of natural philoſophy, therefore, in which the circumſtances of the phenomena can be properly expreſſed by numbers, or geometrical magnitudes, algebra may be employed, both in promoting the inveſtigation of phyſical laws by experience, and alſo in deducing the neceſſary conſequences of laws inveſtigated, and preſumed to be juſt.

It is to be obſerved likewiſe, that if various hypotheſes be aſſumed, concerning phyſical quantities, without regard to what takes place in nature, their conſequences may be demonſtratively deduced; and thus a ſcience may be eſtabliſhed, which may be properly called *mathematical*. The uſe of algebra in this ſcience, which is ſometimes called *Theoretical Mechanics*, is obvious from the principles already laid down.

In conducting theſe inquiries, it is to be obſerved, that, for the ſake of brevity, the language of algebraical operations is often uſed, with regard to phyſical quantities themſelves; though it is always to be underſtood, that, in ſtrict propriety, it can be applied only to the mathematical notations of theſe quantities.

Before illuſtrating this application of algebra by examples, it may be proper to explain a method of ſtating the proportion of variable quantities, and reaſoning with regard to it, which is of general uſe in natural philoſophy.

1. *Of the proportion of variable quantities.*

Mathematical quantities are often ſo connected, that when the magnitude of one is varied, the magnitudes of the others are varied, according to a determined rule. Thus, if two ſtraight lines, given in poſition, interſect each other; and, if a ſtraight line, cutting both, moves parallel to itſelf, the two ſegments of the given lines between their interſection, and the moving line, however varied, will always have the ſame proportion. Thus alſo, if an ordinate to the diameter of a parabola move parallel to itſelf, the abſciſs will be increaſed or diminiſhed, in proportion as the ſquare of the ordinate is increaſed or diminiſhed.

In like manner may algebraical quantities be connected. If x, y, z, &c. repreſent any variable quantities, while a, b, c, repreſent ſuch as are conſtant or invariable, then an equation containing two or more variable quantities, with any number of conſtant quantities, will exhibit a relation of variable quantities, ſimilar to thoſe already mentioned. Thus, if $ax=by$, then $x:y::b:a$, that is, x has a conſtant proportion to y, in whatever way theſe two quantities may be varied. Likewiſe, if $xy^2=a^2b$, then $y^2:a^2::b:x$, or $y^2:\frac{1}{x}::a^2:\frac{1}{b}$, that is, y^2 has a conſtant proportion to the reciprocal of x, or y^2 is increaſed in the ſame proportion as

x is diminiſhed, and converſely. It is neceſſary to premiſe the following definitions.

DEFINITIONS.

Let there be any number of variable quantities X, Y, Z, V, &c. connected in ſuch a manner, that, when X becomes x, Y, Z, V, &c. become reſpectively y, z, v, &c. And let a, b, c, &c. repreſent any conſtant quantities, whether given or unknown. Then,

1. If two variable quantities X and Y are ſo connected, that, whatever be the values of x and y, $X : x :: Y : y$, this proportion is expreſſed thus, $X = Y$, and X is ſaid to be *directly as* Y, or ſhortly, X is ſaid to be *as* Y.

2. If two variable quantities X and Y are ſo connected, that $X : x :: y : Y$, or $X : x :: \frac{1}{Y} : \frac{1}{y}$; their relation is thus expreſſed, $X = \frac{1}{Y}$; and X is ſaid to be *inverſely*, or *reciprocally as* Y.

3. If X, Y, Z, are three variable quantities, ſo connected that $X : x :: YZ : yz$, their

relation is ſo expreſſed, $X=YZ$, and X is ſaid to be *directly as* Y and Z, *jointly*; or X is ſaid to be *as* Y and Z.

4. If any number of variable quantities as X, Y, Z, V, &c. are ſo connected, that $XY : xy :: \frac{YZ}{V} : \frac{yz}{v}$; then $XY=\frac{YZ}{V}$, and XY is ſaid to be *directly as* YZ, and *inverſely as* V, or more explicitly, X and Y *jointly*, are *directly as* Y and Z *jointly*, and *inverſely as* V.

In like manner are other combinations of variable qualities denoted and expreſſed.

It is to be obſerved alſo, the ſame definitions take place, when the variable quantities are multiplied or divided by any conſtant quantities. Thus, if $aX : ax :: \frac{b}{Y} : \frac{b}{y}$, then $ax=\frac{b}{Y}$, &c.

5. Let the preceding notation of proportion be called a *proportional equation* *, the

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* Theſe terms are uſed only with a view to give more preciſion to the ideas of beginners. In order to avoid the ambiguity in the meaning of the ſign $=$, ſome writers employ the character $\propto$ to denote conſtant proportion; but this is ſeldom neceſſary, as the quantities compared are generally of different kinds, and the relation expreſſed is ſufficiently obvious. See Emerſon's Mathematics, Vol. 1.

equations formerly treated of being in this place, for the ſake of diſtinction, called *abſolute.*

Cor. Every abſolute equation, containing more than one variable quantity, may be conſidered as a proportional equation; and, in a proportional equation, if at any particular correſponding values of the variable quantities, the equation becomes abſolute, it will be univerſally abſolute.

Prop. I. If one ſide of a proportional equation be either multiplied or divided by any conſtant quantity, it will continue to be true. Thus, if $X = \frac{1}{Y}$, then $X = \frac{a^2}{bY}$. For, ſince $X = \frac{1}{Y}$ (Def. 3.) $X : x :: \frac{1}{Y} : \frac{1}{y}$, it follows, (Chap. 2.), that $X : x :: \frac{a^2}{bY} : \frac{a^2}{by}$, therefore (Def. 4.) $X = \frac{a^2}{bY}$.

Prop. 2. If the two ſides of a proportional equation be both multiplied, or both divided by the ſame quantity, it will continue to be true.

1*ſt*, If the quantity be conſtant, it is manifeſt from Prop. 1.

2d, If the quantity be variable, let $X=Y$, and Z a variable quantity, then $XZ=YZ$. For, ſince $X=Y$, (Def. 2.) $X:x::Y:y$; multiply the antecedents by Z, and the conſequents by z, then $XZ:xz::YZ:yz$, therefore (Def. 5.) $XZ=YZ$. In like manner, if $X=Y$, $\frac{X}{Z}=\frac{Y}{Z}$.

Cor. Any variable quantity, which is a factor of one ſide of a proportional equation, may be made to ſtand alone. Thus, if $XY=\frac{Z}{V}$, then $X=\frac{Z}{VY}$; alſo, $Z=XYV$; and $Y=\frac{Z}{XV}$, and alſo $V=\frac{Z}{XY}$, &c. Hence, alſo, if one ſide of a proportional equation be divided by the other, the quotient is a conſtant quantity, viz. 1.

Prop. 3. If two proportional equations have a common ſide, the remaining two ſides will form a proportional equation. Alſo, that common ſide will be *as* the ſum or difference of the other two.

Thus, if $X=Y$, and $Y=Z$, then $X=Z$. For $X:x::Y:y$, and $Y:y::Z:z$, therefore multiplying theſe ratios, $XY:xy::YZ$

$: yz$, and by dividing antecedents and conſequents, $X : x :: Z : z$, therefore (Def. 1.) $X=Z$.

Likewiſe, if $X=Y$, and $Y=Z$, $Y=X\pm Z$. For, ſince $X : x :: Y : y :: Z : z$. (Chap. 2.) $Y : y :: X\pm Z : x\pm z$, therefore (Def. 5.) $Y=X\pm Z$.

Cor. Hence, one ſide of a proportional equation will be *as* the ſum, or *as* the difference of the two ſides; and the ſum of the two ſides will be *as* their difference. Thus, if $X=Y+Z$, then $\mathrm{X}=X+Y+Z$ and $X=X-Y-Z$, and alſo $X+Y+Z=X-Y-Z$.

Prop. 4. If the two ſides of a proportional equation be reſpectively multiplied or divided by the two ſides of any other proportional equation, the products or quotients will form a proportional equation.

Thus, if $X=Y$, and $Z=V$, then $XZ=YV$. For, ſince $X : x :: Y : y$, and $Z : z :: V : v$, by multiplying theſe proportions (Chap. 1. 2.) $XZ : xz :: YV : yv$, therefore (Def. 5.) $XZ=YV$. In like manner in the caſe of diviſion.

Cor. 1. The two ſides of a proportional equation may be raiſed to any power, or, any root may be extracted out of both, and the equation will continue to be true.

Thus, if $X=Y$, then $X^m=Y^m$, for, ſince $X=Y$, $X : x :: Y : y$, and therefore $X^m : x^m :: Y^m : y^m$; therefore $X^m=Y^m$. And, if $X=Y$, alſo $X^{\frac{1}{m}}=Y^{\frac{1}{m}}$.

Cor. 2. If two proportional equations have a common ſide, that ſide will be *as* the ſquare root of the product of the other two. Thus, if $X=Y$, and $Y=Z$, by this prop. $Y^2=XZ$, and (Cor. 1.) $Y=\sqrt{XZ}$. Hence alſo, in this caſe, $\sqrt{XZ}=X\pm Z$; for (prop. 3.) $Y=X\pm Z$.

Cor. 3. If one ſide of a proportional equation be a factor of a ſide of another proportional equation, the remaining ſide of the former may be inſerted in the latter, in place of that factor. Thus, if $X=ZY$, and $Z=\frac{1}{V}$, then $X=\frac{Y}{V}$, as appears by multiplying the two equations, and dividing by Z.

Prop. 5. Any proportional equation may be made abſolute, by multiplying one ſide by a conſtant quantity.

Thus, if $X = Y$, then let two particular correſponding values of theſe variable quantities be aſſumed as conſtant, and let them be a and b, then $X : a :: Y : b$, and $Xb = aY$, or $X = Y \times \frac{a}{b}$, an abſolute equation.

SCHOLIUM.

1. If there be two variable phyſical quantities, either of the ſame, or of different kinds, which are ſo connected, that, when the one is increaſed or diminiſhed, the other is increaſed or diminiſhed in the ſame proportion; or, if the magnitudes of the one, in any two ſituations, have the ſame ratio to each other, as the magnitudes of the other in the correſponding ſituations, the relation of the mathematical meaſures of theſe quantities may be expreſſed by a proportional equation, according to Def. 1.

2. If two variable phyſical quantities be ſo connected, that the one increaſes in the ſame proportion as the other is diminiſhed, and converſely; or, if the magnitudes of the one, in any two ſituations, be reciprocally proportioned to the magnitudes of the other,

in the correſponding ſituations, the relation of their meaſures may be expreſſed by a proportional equation, according to Def. 2.

3. If three variable phyſical quantities are ſo connected, that one of them is increaſed or diminiſhed, in proportion as both the others are increaſed or diminiſhed; or, if the magnitudes of one of them in any two ſituations, have a ratio, which is compounded of the ratios of the magnitudes of the other two, in the correſponding ſituations; the relation of the meaſures of theſe three may be expreſſed by a proportional equation, according to Def. 3.

4. In like manner may the relations of other combinations of phyſical quantities be expreſſed, according to Def. 4. And when theſe proportional equations are obtained, by reaſoning with regard to them, according to the preceding propoſitions, new relations of the phyſical quantities may be deduced.

2. *Examples of Phyſical Problems.*

The uſe of algebra, in natural philoſophy, may be properly illuſtrated by ſome examples of phyſical problems. The ſolution of ſuch problems muſt be derived from known phyſical laws, which, though ultimately founded on experience, are here aſſumed as principles, and reaſoned upon mathematically. The experiments by which the principles are aſcertained admit of various degrees of accuracy; and on the degree of phyſical accuracy in the principles will depend the phyſical accuracy of the concluſions mathematically deduced from them. If the principles are inaccurate, the concluſions muſt, in like manner, be innacurate; and, if the limits of inaccuracy in the principles can be aſcertained, the correſponding limits, in the concluſions derived from them, may likewiſe be calculated.

EXAMPLE I.

Let a glaſs tube, 30 *inches* (a) *long, be filled with mercury, excepting* 8 *inches* (b); *and let it be inverted, as in the Torricellian experiment, ſo that the* 8 *inches of common air may riſe to the top: It is required to find at what height the mercury will remain ſuſpended, the mercury in the barometer*

barometer being at that time 28 *inches* (d) *high.*

The ſolution of this problem depends upon the following principles:

1. The preſſure of the atmoſphere is meaſured by the column of mercury in the barometer; and the elaſtic force of the air, in its natural ſtate, which reſiſts this preſſure, is therefore meaſured by the ſame column.

2. In different ſtates, the elaſtic force of the air is reciprocally as the ſpaces which it occupies.

3. In this experiment, the mercury which remains ſuſpended in the tube, together with the elaſtic force of the air in the top of it, being a counterbalance to the preſſure of the atmoſphere, may therefore be expreſſed by the column of mercury in the barometer.

Let the mercury in the tube be x inches, the air in the top of it occupies now the ſpace $a-x$; it occupied formerly b inches, and its elaſtic force was d inches of mercury: Now, therefore, the force muſt be

$(a-x : b :: d :)\ \frac{bd}{a-x}$ inches (2.). Therefore (3.) $x+\frac{bd}{a-x}=d$. This reduced, and putting $a+d=2m$, the equation is $x^2-2mx=bd-ad$.

This reſolved gives $x=m\pm\sqrt{m^2+bd-ad}$.
In numbers - - $x=44$ or 14.

One of the roots 44 is plainly excluded in this caſe, and the other 14 is the true anſwer. If the column of mercury x, ſuſpended in the tube, were a counterbalance to the preſſure of the atmoſphere, expreſſed by the height of the barometer d, together with the meaſure of the elaſtic force of b inches of common air in the ſpace $x-a$, that is, if $x=d+\frac{bd}{x-a}$, or $x-\frac{bd}{x-a}=d$, the equation will be the ſame as before, and the root 44 would be the true anſwer. But the experiment in this queſtion does not admit of ſuch a ſuppoſition.

EXAMPLE II.

The diſtance of the earth and moon (d,) *and their quantities of matter* (t, l,) *being given, to find the point of equal attraction, between them.*

Let the diſtance of the point from the earth be x, its diſtance from the moon will be therefore $d-x$. But gravitation is *as* the matter *directly*, and *as* the ſquare of the diſtance *inverſely*; therefore the earth's attraction is *as* $\frac{t}{x^2}$; and the moon's attraction is *as* $\frac{l}{\overline{d-x}|^{2}}$. But theſe are here equal; therefore,

$$\frac{t}{x^2}=\frac{l}{\overline{d-x}|^{2}} \text{ and } \frac{\sqrt{t}}{x}=\frac{\sqrt{l}}{d-x}$$

This equation reduced gives $x=\frac{d\sqrt{t}}{\sqrt{t}+\sqrt{l}}$

$$\left.\begin{array}{l}\text{Or mult. numerator and de-}\\ \quad\text{nominator by } \sqrt{t}-\sqrt{l}\end{array}\right\} x=\frac{dt-d\sqrt{tl}}{t-l}.$$

In round numbers, let $d=60$ ſemidiameters of the earth, $t=40$, $l=1$, then $x=52$ ſemidiameters nearly. There is another point beyond the moon at which the attractions are equal, and it would be found by putting the ſquare root of $\overline{d-x}|^{2}$ to be $x-d$, which, in this caſe, would be a poſitive quantity; and then $x=\frac{dt+d\sqrt{tl}}{t-l}=72$ nearly. If the quantities had been multiplied before extracting the ſquare roots, the adfected quadratic would have given the ſame two roots.

EXAMPLE III.

Let a ſtone be dropt into an empty pit; and let the time from the dropping of it to the hearing the ſound from the bottom be given: To find the depth of the pit.

Let the given time be a; let the fall of a heavy body in the 1ſt ſecond of time (16.122 feet) be b; alſo, let the motion of ſound in a ſecond (1142 feet) be c.

Let the time of the ſtone's fall be	1	x
The time in which the ſound of it moves to the top is	2	$a-x$
The deſcent of a falling body is as the ſquare of the time, therefore the depth of the pit is ($1^2 : x^2 :: b :$)	3	bx^2
The depth from the motion of ſound is alſo	4	$ca-cx$
Therefore 3 and 4	5	$bx^2=ca-cx$

This equation being reſolved, gives the value of x, and from it may be got bx^2 or $ca-cx$, the depth of the pit.

If the time is 10″, then $x=8.885$ nearly, and the depth is 1273 feet.

There are ſeveral circumſtances in this problem which render the concluſion inaccurate.

1. The values of c and b, on which the ſolution is founded, are derived from experiments, which are ſubject to conſiderable inaccuracies.

2. The reſiſtance of the air has a great effect in retarding the deſcent of heavy bodies, when the velocity becomes ſo great as is ſuppoſed in this queſtion; and this circumſtance is not regarded in the ſolution.

3. A ſmall error, in making the experiment to which this queſtion relates, produces a great error in the concluſion. This circumſtance is particularly to be attended to in all phyſical problems; and, in the preſent caſe, without noticing the preceding imperfections, an error of half a ſecond, in eſtimating the time, makes an error of above 100 feet in the expreſſion of the depth of the pit.

III. *Of Intereſt and Annuities.*

The application of algebra to the calculation of intereſts and annuities, will furniſh proper examples of its uſe in buſineſs. Algebra cannot determine the propriety or juſtice of the common ſuppoſition on which theſe calculations are founded, but only the neceſſary concluſions reſulting from them.

Notation.

In the following theorems, let p denote any principal ſum of which 1 l. is the unit, t the time during which it bears intereſt, of which 1 year ſhall be the unit, r the rate of intereſt of 1 l. for 1 year, and let s be the amount of the principal ſum p with its intereſt, for the time t, at the rate r.

I. *Of Simple Intereſt.*

$s=p+ptr$, and of theſe four, s, p, t, r, any three being given, the fourth may be found by reſolving a ſimple equation.

The foundation of the canon is very obvious; for the intereſt of 1 l. in one year is r, for t years it is tr, and for p pounds it is ptr; the whole amount of principal and intereſt muſt therefore be $p+ptr=s$.

II. *Of Compound Intereſt.*

When the ſimple intereſt at the end of every year is ſuppoſed to be joined to the principal ſum, and both to bear intereſt for the following year, money is ſaid to bear compound intereſt. The ſame notation being uſed, let $1+r=R$. Then $s=pR^t$.

For the ſimple intereſt of 1 l. in a year is r, and the new principal ſum therefore which bears intereſt during the ſecond year is $1+r=R$; the intereſt of R for a year is rR, and the amount of principal and intereſt at the end of the 2d year is $R+rR=R\times\overline{1+r}=R^2$. In like manner, at the end of the 3d year it is R^3, and at the end of t years it is R^t, and for the ſum p it is $pR^t=s$.

Cor. 1. Of theſe four p, R, t, s, any three being given, the fourth may be found.

When

When t is not very ſmall, the ſolution will be obtained moſt conveniently by logarithms. When R is known r may be found, and converſely.

Ex. If 500 l. has been at intereſt for 21 years, the whole arrear due, reckoning $4\frac{1}{2}$ *per cent.* compound intereſt, is 1260.12 l. or 1260 l. 2 s. 5 d. In this caſe, $p=500$, $R=1.045$, and $t=21$, and $s=1260.12$, and any one of theſe may be derived by the theorem from the others being known. Thus, to find s; $l.R^t=t\times l.R=21\times 0.0191163$ $=0.4014423$, therefore $R^t=2.52042$, and $s=(pR^t=)\ 500\times 2.520242=1260.121$.

Cor. 2. The preſent worth of a ſum (s) in reverſion that is payable after a certain time t is found thus. Let the preſent worth be x, then this money improved by compound intereſt during t produces xR^t, which muſt be equal to s, and if $xR^t=s$, $x=\frac{s}{R^t}$.

Cor. 3. The time in which a ſum is doubled at compound intereſt will be found thus, $pR^t=2p$, and $R^t=2$, and $t=\frac{l.2.}{l.R.}$; thus, if the rate is 5 *per cent.* $r=.05$, and

$\frac{l.2}{l.1.05} = \frac{0.3010300}{0.0211893} = 14.2066$; that is 14 years and 75 days nearly.

SCHOLIUM.

Many other ſuppoſitions might be made with regard to the improvement of money by compound intereſt. The intereſt might be ſuppoſed to be joined to the capital, and along with it to bear intereſt at the end of every month, at the end of every day, or even at the end of every inſtant, and ſuitable calculations might be formed; but theſe ſuppoſitions, being ſeldom uſed in practice, are omitted.

III. *Of Annuities.*

An annuity is a payment made annually for a certain term of years, and the chief problem, with regard to it is, ‘ to determine

‘ its preſent worth.’ The ſuppoſition on which the ſolution proceeds is, that the money received by the ſeller, being improved by him in a certain manner during the continuance of the annuity, amounts to the ſame ſum as the ſeveral payments received by the purchaſer, improved in the ſame manner. The ſuppoſitions with regard to the improvement may be various. What is called the method of ſimple intereſt, in which ſimple intereſt only is reckoned upon the purchaſe-money, and ſimple intereſt on each annuity from the time of payment, is ſo manifeſtly unequitable, as to be univerſally rejected; and the ſuppoſition which is now generally admitted in practice, is the higheſt improvement poſſible on both ſides, viz. by compound intereſt. As the taking compound intereſt is prohibited by law, the realizing of this ſuppoſed improvement requires punctual payment of intereſt, and therefore the intereſt in ſuch calculations is uſually made low. Even with this advantage, it can hardly be rendered effectual in its full extent; it is however univerſally acquieſced in, as the moſt proper foundation of general rules; and when peculiar circumſtances require any different hypotheſis, a ſuitable calculation may be made.

Let then the annuity be called a, and let p be the preſent worth of it or purchaſe-money, t the time of its continuance, and let the other letters denote as formerly.

The ſeller, by improving the price received p, at compound intereſt, at the time the annuity ceaſes, has pR^t.

The purchaſer is ſuppoſed to receive the firſt annuity a at the end of the firſt year, which is improved by him for $t-1$ years; it becomes therefore (Th. 2.) aR^{t-1}.

He receives the 2d annuity at the end of the 2d year, and when improved $t-2$, it becomes aR^{t-2}.

The 3d annuity becomes aR^{t-3}, &c.

The laſt annuity is ſimply a, therefore the whole amount of the improved annuities is the geometrical ſeries $a+aR+aR^2$, &c. . . . aR^{t-1}. The ſum of the ſeries by Chap. 6. Sect. 2. is $a\times\frac{R^t-1}{R-1}=a\times\frac{R^t-1}{r}$.

But, from the nature of the problem, $pR^t = a \times \frac{R^t - 1}{r}$, and hence $p = a \times \frac{R^t - 1}{rR^t}$

$$= a \times \frac{1 - \frac{1}{R^t}}{r}.$$

The ſame concluſion reſults from calculating the preſent worth of the ſeveral annuities, conſidered as ſums payable in reverſion.

Cor. 1. Of theſe four p, a, R, t, any three being given, the fourth may be found, by the ſolution of equations; t is found eaſily by logarithms, R or r can be found only by reſolving an adfected equation of the t order.

Cor. 2. If an annuity has been unpaid for the term t, the arrear, reckoning compound intereſt, will be $a \times \frac{R^t - 1}{r}$.

Cor. 3. The preſent worth of an annuity in reverſion, that is to commence after a certain time (n), and then to continue t

years, is found by ſubtracting the preſent worth for n years from the preſent worth for $n+t$ years, and then,

$$p=a\times\frac{R^t-1}{rR^{t+n}}=a\times\frac{1-\frac{1}{R^t}}{rR^n}$$

Alſo of R, t, n, a, p, any four being given, the fifth may be found.

Cor. 4. If the annuity is to continue forever, then R^t-1 and R^t may be conſidered as the ſame ; and $p=a\times\frac{R^t-1}{rR^t}=\frac{a}{r}$.

Cor. 5. A perpetuity in reverſion (by Cor. 3.) ſince $R^t-1=R^t$, is $p=\frac{a}{rR^n}$.

Prob. When 12 years of a leaſe of 21 were expired, a renewal for the ſame term was granted for 1000 l. ; 8 years are now expired, and for what ſum muſt a correſponding renewal be made, reckoning 5 *per cent.* compound intereſt ?

From the firſt tranſaction the yearly profit rent muſt be deduced ; and from this the proper fine in the ſecond may be computed.

In the firſt bargain, an annuity in reverſion for 12 years, to commence 9 years hence, was ſold for 1000 l. the annuity will therefore be found by Cor. 3. in which all the quantities are given, but $a=p\times\frac{rR^n}{1-\frac{1}{R^t}}$.

and by inſerting numbers, viz. $p=1000$, $t=12$, $n=9$, $r=.05$, and $R=1.05$; and working by logarithms $a=175.029=175$ l. 7 d.

Next, having found a, the ſecond renewal is made by finding the preſent worth of the annuity a in reverſion, to commence 13 years hence, and to laſt 8 years. In the canon (Cor. 3.) inſert for a, 175.029, and let $t=8$, $n=13$, and $r=.05$ as before, $p=599.93=599$ l. 18 s. 6½ d. The fine required.

As theſe computations often become troubleſome, and are of frequent uſe, all the common caſes are calculated in tables, from which the value of any annuity, for any time, at any intereſt, may eaſily be found.

It is to be obſerved alſo, that the preceding rules are computed on the ſuppoſition of the annuities being paid yearly ; and therefore, if they be ſuppoſed to be paid half yearly, or quarterly, the concluſions will be ſomewhat different, but they may eaſily be calculated on the preceding principles.

The calculations of life annuities depend partly upon the principles now explained, and partly on phyſical principles, from the probable duration of human life, as deduced from bills of mortality.

ELEMENTS

OF

ALGEBRA.

PART II.

Of the General Properties and Reſolution of EQUATIONS of all Orders.

CHAP. I.

Of the Origin and Compoſition of Equations; and of the Signs and Coefficients of their Terms.

IN order to reſolve the higher orders of equations, and to inveſtigate their general affections, it is proper firſt to conſider their origin from the combination of inferior equations.

As it would be impoſſible to exhibit particular rules for the ſolution of every order of equations, their number being indefinite; there is a neceſſity of deducing rules from their general properties, which may be equally applicable to all.

In the application of algebra to certain ſubjects, and eſpecially to geometry, there may be an oppoſition in the quantities, analogous to that of addition and ſubtraction, which may therefore be expreſſed by the ſigns + and —. Hence theſe ſigns may be underſtood by abſtraction, to denote contrariety in general; and therefore, in this method of treating of equations, negative roots are admitted, as well as poſitive. In many caſes the negative will have a proper and determinate meaning; and when the equation relates to magnitude only, where contrariety cannot be ſuppoſed to exiſt, theſe roots are neglected, as in the caſe of quadratic equations formerly explained. There is beſides this advantage in admitting negative roots, that both the properties of equations from which their reſolution is obtained, and alſo thoſe which are uſeful in the many extenſive applications of algebra, become more ſimple and general, and are more eaſily deduced.

In this general method, all the terms of any equation are brought to one ſide, and the equation is expreſſed by making them equal to 0. Therefore, if a root of the equation be inſerted inſtead of (x) the unknown quantity, the poſitive terms will be equal to the negative, and the whole muſt be equal to 0.

DEV. When any equation is put into this form, the term in which (x) the unknown quantity is of the higheſt power, is called the *Firſt*, that in which the index of x is leſs by 1, is the *Second*, and ſo on, till the laſt into which the unknown quantity does not enter, and which is called the *Abſolute* Term.

Prop. I. If any number of equations be multiplied together, an equation will be produced, of which the dimenſion * is equal to the ſum of the dimenſions of the equations multiplied.

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* The term *dimenſion*, in this treatiſe, is uſed in ſenſes ſomewhat different, but ſo as not to create any ambiguity. In this chapter it means either the order of an equation, or the number denoting that order, which was formerly defined to be the higheſt exponent

If any number of ſimple equations be multiplied together, as $x-a=0$, $x-b=0$, $x-c=0$, &c. it is obvious, that the product will be an equation of a dimenſion containing as many units as there are ſimple equations. In like manner, if higher equations are multiplied together, as a cubic and a quadratic, one of the fifth order is produced; and ſo on.

Converſely. An equation of any dimenſion is conſidered as compounded either of ſimple equations, or of others, ſuch that the ſum of their dimenſions is equal to the dimenſion of the given one. By the reſolution of equations theſe inferior equations are diſcovered, and by inveſtigating the component ſimple equations, the roots of any higher equation are found.

Cor. 1. Any equation admits of as many ſolutions, or has as many roots, as there are ſimple equations which compoſe it, that is, as there are units in the dimenſion of it.

Cor.

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nent of the unknown quantity in any term of the equation.

Cor. 2. And converſely, no equation can have more roots than the units in its dimenſion.

Cor. 3. Imaginary or impoſſible roots muſt enter an equation by pairs; for they ariſe from quadratics, in which both the roots are ſuch.

Hence alſo, an equation of an even dimenſion may have all its roots, or any even number of them impoſſible, but an equation of an odd dimenſion muſt at leaſt have one poſſible root.

Cor. 4. The roots are either poſitive or negative, according as the roots of the ſimple equations, from which they are produced, are poſitive or negative.

Cor. 5. When one root of an equation is diſcovered, one of the ſimple equations is found, from which the given one is compounded. The given equation, therefore, being divided by this ſimple equation, will give an equation of a dimenſion lower by 1. Thus, any equation may be depreſſed as many degrees as there are roots found by any method whatever.

Prop. II. To explain the general properties of the ſigns and coefficients of the terms of an equation.

Let $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, &c. be ſimple equations, of which the roots are any poſitive quantities $+a$, $+b$, $+c$, $+d$, &c. and let $x+m=0$, $x+n=0$, &c. be ſimple equations, of which the roots are any negative quantities $-m$, $-n$, &c. and let any number of theſe equations be multiplied together, as in the following table.

$$x-a=0$$
$$\times\overline{x-b}=0$$
$$\overline{\hphantom{xxxxxxxxxxxx}}$$
$$=\left.\begin{matrix}x^2-ax\\ \quad -bx+ab\end{matrix}\right\}=0,\text{ a Quadratic.}$$
$$\times\overline{x-c}=0$$
$$\overline{\hphantom{xxxxxxxxxxxxxxxx}}$$
$$=x^3\left.\begin{matrix}-a\\-b\\-c\end{matrix}\right\}\times x^2\left.\begin{matrix}+ab\\+ac\\+bc\end{matrix}\right\}\times x-abc=0,\text{ a Cubic.}$$
$$\times\overline{x+m}=0$$
$$\overline{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxx}}$$
$$=x^4\left.\begin{matrix}-a\\-b\\-c\\+m\end{matrix}\right\}\times x^3\left.\begin{matrix}+ab\\+ac\\+bc\\-am\\-bm\\-cm\end{matrix}\right\}\times x^2\left.\begin{matrix}-abc\\+abm\\+acm\\+bcm\end{matrix}\right\}\times x-abcm=0,\text{ a Biquadratic, \&c.}$$

From this table it is plain,

1. That in a complete equation the number of terms is always greater by unit than the dimenſion of the equation.

2. The coefficient of the firſt term is 1. The coefficient of the ſecond term is the ſum of all the roots *a*, *b*, *c*, *m*, &c. with their ſigns changed.

The coefficient of the third term is the ſum of all the products that can be made by multiplying any two of the roots together.

The coefficient of the fourth term is the ſum of all the products which can be made by multiplying together any three of the roots with their ſigns changed; and ſo of others.

The laſt term is the product of all the roots, with their ſigns changed.

3. From induction it appears, that in any equation (the terms being regularly arranged as in the preceding example) there are as many poſitive roots as there are changes in the ſigns of the terms from + to —,

and from — to +; and the remaining roots are negative. The rule alſo may be demonſtrated.

Note. The impoſſible roots in this rule are ſuppoſed to be either poſitive or negative.

In this example of a numeral equation $x^4-10x^3+35x^2-50x+24=0$, the roots are, $+1$, $+2$, $+3$, $+4$, and the preceding obſervations with regard to the ſigns and coefficients take place.

Cor. If a term of an equation is wanting, the poſitive and negative parts of its coefficient muſt then be equal. If there is no abſolute term, then ſome of the roots muſt be $=0$, and the equation may be depreſſed by dividing all the terms by the loweſt power of the unknown quantity in any of them. In this caſe alſo, $x-0=0$, $x-0=0$, &c. may be conſidered as ſo many of the component ſimple equations, by which the given equation being divided, it will be depreſſed ſo many degrees.

CHAP.

CHAP. II.

Of the Transformation of Equations.

THERE are certain transformations of equations neceſſary towards their ſolution; and the moſt uſeful are contained in the following propoſitions.

Prop. I. The affirmative roots of an equation become negative, and the negative become affirmative, by changing the ſigns of the alternate terms, beginning with the ſecond.

Thus the roots of the equation $x^4-x^3-19x^2+49x-30=0$ are $+1, +2, +3, -5$, whereas the roots of the equation, $x^4+x^3-19x^2-49x-30=0$, are $-1, -2, -3, +5$.

The reaſon of this is derived from the compoſition of the coefficients of theſe terms, which conſiſt of combinations of odd numbers of the roots, as explained in the preceding chapter.

Prop. II. An equation may be tranſformed into another that ſhall have its roots greater or leſs than the roots of the given equation by ſome given difference.

Let x be the unknown quantity of the equation, and e the given difference; let $y=x\pm e$, then $x=y\mp e$; and if for x and its powers in the given equation, $y\mp e$ and its powers be inſerted, a new equation will ariſe, in which the unknown quantity is y, and its value will be $x\pm e$; that is, its roots will differ from the roots of the given equation by e.

Let the equation propoſed be $x^3-px^2+qx-r=0$, of which the roots muſt be diminiſhed by e. By inſerting for x and its powers, $y+e$ and its powers, the equation required is,

$$\left.\begin{array}{rrrr} y^3+3ey^2 & +3e^2y & + & e^3 \\ -py^2 & -2pey & - & pe^2 \\ & +qy & + & qe \\ & & - & r \end{array}\right\}=0.$$

Cor. 1. From this transformation, the ſecond, or any other intermediate term, may be taken away; granting the reſolution of equations.

Since the coefficients of all the terms of the transformed equation, except the firſt, involve the powers of e and known quantities only, by putting the coefficient of any term equal to 0, and reſolving that equation, a value of e may be determined; which being ſubſtituted, will make that term to vaniſh.

Thus, in this example, to take away the ſecond term, let its coefficient, $3e-p=0$, and $e=\frac{1}{3}p$, which being ſubſtituted for e, the new equation will want the ſecond term. And univerſally, the coefficient of the firſt term of a cubic equation being 1, and x being the unknown quantity, the ſecond term may be taken away, by ſuppoſing $x=y\mp\frac{1}{3}p$, $\pm p$ being the coefficient of that term.

Cor. 2. The ſecond term may be taken away by the ſolution of a ſimple equation, the third by the ſolution of a quadratic, and ſo on.

Cor. 3. If the ſecond term of a quadratic equation be taken away, it will become a pure equation, and thus a ſolution of quadratics will be obtained, which coincides with the ſolution already given in Part I.

Cor. 4. The laſt term of the transformed equation is the ſame with the given equation, only having e in place of x.

Prop. III. In like manner may an equation be transformed into another, of which the roots ſhall be equal to the roots of the given equation, multiplied or divided by a given quantity.

Let x be the unknown letter in the given equation, and y that of the equation wanted; alſo let e be the given quantity.

To multiply the roots, let $xe=y$, and $x=\frac{y}{e}$.

To divide the roots let $\frac{x}{e}=y$, and $x=ye$.

Then ſubſtitute for x and its powers, $\frac{y}{e}$ or ye and its powers, and the new equation of which y is the unknown quantity, will have the property required.

Cor. 1. By this propoſition an equation, in which the coefficient of the firſt term is any known quantity, as a, may be tranſformed into another, in which the coefficient of the firſt term ſhall be unit. Thus, let the equation be $ax^3 - px^2 + qx - r = 0$. Suppoſe $y = ax$, or $x = \frac{y}{a}$, and for x and its powers inſert $\frac{y}{a}$ and its powers, and the equation becomes $\frac{y^3}{a^2} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$, or $y^3 - py^2 + qay - a^2r = 0$. Alſo, let the equation be $5x^3 - 6x^2 + 7x - 30 = 0$; and if $x = \frac{y}{5}$, then $y^3 - 6y^2 + 35y - 750 = 0$.

Cor. 2. If the two transformations in Prop. 2. and 3. be both required, they may be performed either ſeparately or together.

Thus, if it is required to transform the equation $ax^3 - px^2 + qx - r = 0$ into one which ſhall want the ſecond term, and in which the coefficient of the firſt term ſhall be 1; let $x=\frac{y}{a}$, and then $y^3-py^2+qay-a^2r=0$ as before; then let $y=z+\frac{1}{3}p$, and the new equation, of which z is the unknown quantity, will want the ſecond term, and the coefficient of z^3, the higheſt term is 1. Or, if $x=\frac{z+\frac{1}{3}p}{a}$, the ſame equation as the laſt found will ariſe from one operation.

Ex. Let the equation be $5x^3-6x^2+7x-30=0$. If $x=\frac{y}{5}$, then $y^3-6y^2+35y-750=0$. And if $y=z+2$, $z^3+23z-696=0$. Alſo, at once, let $x=\frac{z+2}{5}$, and the equation properly reduced, by bringing all the terms to a common denominator, and then caſting it off, will be $z^3+23z-696=0$, as before.

Cor. 3. If there are fractions in an equation, they may be taken away, by multiplying the equation by the denominators, and by this Prop. the equation may then be transformed into another, without fractions, in which the coefficient of the firſt term is 1. In like manner, may a ſurd coefficient be taken away in certain caſes.

Cor. 4. Hence alſo, if the coefficient of the ſecond term of a cubic equation is not diviſible by 3, the fractions thence ariſing in the transformed equation, wanting the ſecond term, may be taken away by the preceding corollary. But the ſecond term alſo may be taken away, ſo that there ſhall be no ſuch fractions in the transformed equation, by ſuppoſing $x=\frac{z\mp p}{3}$, $\pm p$ being the coefficient of the ſecond term of the given equation. And, if the equation $ax^3-px^2+qx-r=0$ be given, in which p is not diviſible by 3, by ſuppoſing $x=\frac{z+p}{3a}$, the transformed equation reduced is $z^3-\overline{3p^2+9aq}\times z-2p^3+9apq-27a^2r=0$; wanting the ſecond term, having 1 for the coefficient of the firſt term, and the coefficients of the other terms being all integers, the coefficients of the given equation being alſo ſuppoſed integers.

General Corollary to Prop. I. II. III.

If the roots of any of theſe transformed equations be found by any method, the

roots

roots of the original equation, from which they were derived, will eaſily be found from the ſimple equations expreſſing their relation. Thus, if 8 is found to be a root of the transformed equation $z^3+23z-696=0$, (Cor. 2. prop. 3.), ſince $x=\frac{z+2}{5}$, the correſponding root of the given equation, $5x^3-6x^2+7x-30=0$, muſt be $\frac{8+2}{5}=2$. It is to be obſerved alſo, that the reaſoning in Prop. 2. and 3. and the corollaries, may be extended to any order of equations, though in them it is applied chiefly to cubics.

CHAP. III.

Of the Reſolution of Equations.

FROM the preceding principles and operations, rules may be derived for reſolving equations of all orders.

I. CARDAN's *Rule for Cubic Equations.*

The ſecond term of a cubic equation being taken away, and the coefficient of the firſt term being made 1, (by Cor. 1. Prop. 2. and Cor. 1. Prop. 3. Chap. II.) it may be generally repreſented by $x^3 * + 3qx + 2r = 0$; the ſign $+$ in all terms denoting the addition of them, with their proper ſigns. Let $x = m + n$, and alſo $mn = -q$; by the ſubſtitution of theſe values, an equation of the 6th order, but of the quadratic form, is deduced, which gives the values of m and n, and hence,

$$m+n=x=\sqrt[3]{-r+\sqrt{r^2+q^3}}+\sqrt[3]{-r-\sqrt{r^2+q^3}};$$

$$\text{or } x=\sqrt[3]{-r+\sqrt{r^2+q^3}}-\frac{q}{\sqrt[3]{-r+\sqrt{r^2+q^3}}}.$$

Cor. 1. In the given equation, if $3q$ is negative, and if r^2 is leſs than q^3, this expreſſion of the root involves impoſſible quantities; while, at the ſame time, all the roots of that equation are poſſible. The reaſon is, that, in this method of ſolution, it is neceſſary to ſuppoſe that x the root may be divided into two parts, of which the product is q. But it is eaſy to ſhew, that, in this, which is called the *irreducible caſe*, it cannot be done.

For example, the equation, (Ex. 3. Sect. 3. of this chapter), $x^3-156x+560=0$, belongs to the irreducible caſe, and the three roots are $+4$, $+10$, -14, and it is plain that none of theſe roots can be divided into two parts m and n, of which the product can be equal to $-q=\frac{156}{3}=52$; for the greateſt product from the diviſion of the greateſt root -14, is $-7 \times -7 = 49$, leſs than 52.

If the cube root of the compound ſurd can be extracted, the impoſſible parts balance each other, and the true root is obtained.

The geometrical problem of the triſection of an arch is reſolved algebraically, by a cubic equation of this form; and hence the foundation of the rule for reſolving an equation belonging to this caſe, by a table of ſines.

Cor. 2. Biquadratic equations may be reduced to cubics, and may therefore be reſolved by this rule.

Some other claſſes of equations too, may be reſolved by particular rules; but theſe, and every other order of equations, are commonly reſolved by the general rules, which may be equally applied to all.

II. *Solution of Equations, whoſe Roots are commenſurate.*

RULE I.

All the terms of the equation being brought to one ſide, find all the diviſors of the abſolute term, and ſubſtitute them ſucceſſively in the equation for the unknown quantity. That diviſor which, ſubſtituted in this manner, gives the reſult $=0$, ſhall be a root of the equation.

Ex. 1. $\left.\begin{matrix} x^3-3ax^2+2a^2x-2a^2b \\ -bx^2+3abx \end{matrix}\right\}=0.$

The ſimple literal diviſors of $-2a^2b$, are a, b, $2a$, $2b$, any of which may be inſerted for x. Suppoſing $x=+a$, the equation becomes

$\left.\begin{matrix} a^3-3a^3+2a^3-2a^2b \\ -ba^2+3a^2b \end{matrix}\right\}$ which is obviouſly $=0$.

Ex. 2. $x^3-2x^2-33x+90=0.$

The diviſors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90.

The firſt of theſe diviſors, which being inſerted for x, will make the reſult $=0$, is $+3$;

$+3$; $+5$ is another, and it is plain the laſt root muſt be negative, and it is -6.

When 3 is diſcovered to be a root, the given equation may be divided by $x-3=0$, and the reſult will be a quadratic, which being reſolved, will give the other two roots, $+5$ and -6.

The reaſon of the rule appears from the property of the abſolute term formerly defined, viz. that it is the product of all the roots.

To avoid the inconvenience of trying many diviſors, this method is ſhortened by the following.

RULE II.

Subſtitute in place of the unknown quantity ſucceſſively three or more terms of the progreſſion, 1, 0, -1, *&c. and find all the diviſors of the ſums that reſult; then take out all the arithmetical progreſſions that can be found among theſe diviſors whoſe common difference is* 1, *and the values of* x *will*

will be among thoſe terms of the progreſſions which are the diviſors of the reſult ariſing from the ſubſtitution of x = 0. *When the ſeries increaſes, the roots will be poſitive; and when it decreaſes, the roots will be negative.*

EXAMPLE.

Let it be required to find a root of the equation $x^3 - x^2 - 10x + 6 = 0$.

The operation is thus:

Suppoſit.		*Reſult.*	*Diviſors.*	*Ar. pro.*
$x = +1$	- - - - -	-4	1,2,4,	4
$x = 0$	$x^3 - x^2 - 10x + 6 =$	$+6$	1,2,3,6,	3
$x = -1$	- - - - -	$+14$	1,2,7,14,	2

In this example there is only one progreſſion, 4, 3, 2, and therefore 3 is a root, and it is -3, ſince the ſeries decreaſes.

It is evident from the rules for transforming equations, (Chap. 2.), that by inſerting for x, $+1 (= +e)$ the reſult is the abſolute term of an equation, of which the roots are leſs than the roots of the given equation by $1=e$, Cor. 4. Prop. 2. When $x=0$ the reſult is the abſolute term of the given equation. When for x is inſerted $-1 = -e$, the reſult is the abſolute term of an equation whoſe roots exceed the roots of the given equation by $1=e$. Hence, if the terms of the ſeries 1, 0, —1, —2, &c. be inſerted ſucceſſively for x, the reſults will be the abſolute terms of ſo many equations, of which the roots form an increaſing arithmetical ſeries with the difference 1. But, as the commenſurate roots of theſe equations muſt be among the diviſors of their abſolute terms, they muſt alſo be among the arithmetical progreſſions found by this rule. The roots of the given equation, therefore, are to be ſought for among the terms of theſe progreſſions which are diviſors of the reſult, upon the ſuppoſition of $x=0$, becauſe that reſult is its abſolute term.

It is plain that the progreſſion muſt always be increaſing, only it is to be obſerved, that a decreaſing ſeries, with the ſign +, becomes increaſing with the ſign —.

Thus,

Thus, in the preceding example, -4, -3, -2, is an increaſing ſeries, of which -3 is to be tried, and it ſucceeds.

If, from the ſubſtitution of three terms of the progreſſion, 1, 0, -1, &c. there ariſe a number of arithmetical ſerieſes, by ſubſtituting more terms of that progreſſion, ſome of the ſerieſes will break off, and, of courſe, fewer trials will be neceſſary.

III. *Examples of Queſtions producing the higher Equations.*

EXAMPLE I.

It is required to divide 16 *l. between two perſons, ſo that the cube of the one's ſhare may exceed the cube of the other's by* 386.

Let the greater ſhare be x pounds,
And the leſs will be $16-x$;

By the queſtion, $x^3-\overline{16-x}|^3=386$

And by Inv. $2x^3-48x^2+768x-4096=386$

Tranſ. and divide } $x^3-24x^2+384x-2241=0$.

Suppoſ.		*Reſults.*		*Diviſors.*
If $x=+1$;	-	1880	-	1, 2, 4, 5, 8, 10, 20,
$x=0$;	-	2241	-	1, 3, 9, 27, 83.
$x=-1$;	-	2650	-	1, 2, 5, 10, 25, 53.

Where 8, 9, 10, differ by 1; therefore +9 is to be tried; and being inſerted for x, the equation is $=0$. The two ſhares then are 9 and 7 which ſucceed. Since $x=9$; $x-9=0$ is one of the ſimple equations from which this cubic is produced; therefore

$$\frac{x^3-24x^2+384x-2241}{x-9}=x^2-15x+249=0.$$

And the two roots of this quadratic are impoſſible.

EXAMPLE II.

What two numbers are thoſe, whoſe product multiplied by the greater will produce 405, *and their difference multiplied by the leſs* 20 *?*

Let the greater number be x, and the leſs y.

Then by queſt. $\begin{cases} xy\times x=x^2y=405 \\ \overline{x-y}\times y=xy-y^2=20 \end{cases}$

Therefore - - $x=\dfrac{y^2+20}{y}$

And - - - - $x^2=\dfrac{y^4+40y^2+400}{y^2}$

Alſo - - - - $x^2=\dfrac{405}{y}$

Therefore $\dfrac{y^4+40y^2+400}{y^2}=\dfrac{405}{y}$

Mult. and tranſ. $y^4+40y^2-405y+400=0$.

This biquadratic reſolved by diviſors, gives $y=5$; and therefore $x=9$. Alſo, $\frac{y^4+40y^2-405y+400}{y-5}=y^3+5y^2+65y-80=0$.

This cubic equation has one poſitive incommenſurate root, viz. 1.114, &c. which may be found by the rule in the next ſection, and two impoſſible. The incommenſurate root $y=1,114$, &c. gives $x=19.067$, &c. and theſe two anſwer the conditions very nearly.

EXAMPLE III.

The ſum of the ſquares of two numbers 208, *and the ſum of their cubes* 2240 *being given, to find them.*

Let the greater be $x+y$, and the leſs $x-y$.

Then $\overline{x+y}|^2+\overline{x-y}|^2=2x^2+2y^2=208$.

Hence $y^2=104-x^2$.

Alſo $\overline{x+y}|^3+\overline{x-y}|^3=2x^3-6xy^2=2240$.

Subſtitute for y^2 its value and $2x^3+624x-6x^3=2240$.

This reduced gives $x^3-156x+560=0$.

The roots of this equation are $+10, +4, -14$. If $x=10$ then $y=2$, and the numbers ſought are 12 and 8, which give the only juſt ſolution. If $x=4$, then $y^2=88$ and $y=\sqrt{88}$. The numbers ſought are therefore $4+\sqrt{88}$ and $4-\sqrt{88}$. The laſt is negative, but they anſwer the conditions. Laſtly, if $x=-14$, then $y^2=-92$, hence $y=\sqrt{-92}$, is impoſſible; but ſtill the two numbers $-14+\sqrt{-92}$, $-14-\sqrt{-92}$, being inſerted, would anſwer the conditions. But it has been frequently obſerved, that ſuch ſolutions are both uſeleſs, and without meaning.

IV. *Solution of Equations by Approximation.*

By the former rules, the roots of equations when they are commenſurate may be obtained: Theſe, however, more rarely occur; and when they are incommenſurate, we can find only an approximate value of them, but to any degree of exactneſs required. There are various rules for this purpoſe;

purpoſe ; one of the moſt ſimple is that of Sir Iſaac Newton, which ſhall now be explained.

Lemma. If any two numbers being inſerted for the unknown quantity (x) in any equation, give reſults with oppoſite ſigns, an odd number of roots muſt be between theſe numbers.

This appears from the property of the abſolute term, and from this obvious maxim, that, if a number of quantities be multiplied together, and if the ſigns of an odd number of them be changed, the ſign of the product is changed. For, when a poſitive quantity is inſerted for x, the reſult is the abſolute term of an equation whoſe roots are leſs than the roots of the given equation by that quantity, (Prop. 2. Cor. 3. Chap. 2.). If the reſult has the ſame ſign as the given abſolute term, then from the property of the abſolute term, (Prop. 2. Chap. 1.), either none, or an even number only, of the poſitive roots have had their ſigns changed by the transformation ; but, if the reſult has an oppoſite ſign to that of the given abſolute term, the ſigns of an odd number of the poſitive roots muſt have been changed. In the firſt caſe, then, the quantity ſubſtituted muſt have been either greater than each of an even number of the poſitive roots of the given equation, or leſs than any of them; in the ſecond caſe, it muſt have been greater than each of an odd number of the poſitive roots. An odd number of the poſitive roots therefore muſt lie between them, when they give reſults with oppoſite ſigns. The ſame obſervation is to be extended to the ſubſtitution of negative quantities and the negative roots.

From this lemma, by means of trials, it will not be difficult to find the neareſt integer to a root of a given numeral equation. This is the firſt ſtep towards the approximation; and both the manner of continuing it, and the reaſon of the operation, will be evident from the following example.

Let the equation be $x^3-2x-5=0$.

1. Find the neareſt integer to the root; in this caſe a root is between 2 and 3, for

theſe numbers being inſerted for x, the one gives a poſitive, and the other a negative reſult. Either the number above the root, or that below it, may be aſſumed as the firſt value; only, it will be more convenient to take that which appears to be neareſt to the root, as will be manifeſt from the nature of the operation.

2. Suppoſe $x=2+f$, and ſubſtitute this value of x in the equation.

$$\begin{array}{rl} x^3 = & 8+12f+6ff+f^2 \\ -2x = & -4-2f \\ -5 = & -5 \\ \hline x^3-2x-f = & -1+10f+6f^2+f^3=0. \end{array}$$

If f is leſs than unit, its powers f^2 and f^3 may be neglected in this firſt approximation, and $10f=1$, or $f=0.1$ nearly, therefore $x=2.1$ nearly.

3. As $f=0.1$, nearly, let $f=.1+g$, and inſert this value of f in the preceding equation.

$$\begin{array}{rl} f^3 = & 0.001+0.03g+0.3g^2+g^3 \\ 6f^2 = & 0.06+1.2g+6.g^2 \\ 10f = & 1+10g \\ -1 = & -1 \\ \hline f^3+6f^2+10f-1 = & 0.061+11.23g+6.3g^2+g^3=0 \end{array}$$

and neglecting g^2 and g^3 as very ſmall $0.61 + 11.23g = 0$, or $g = \frac{-0.061}{11.23} = -.0054$, hence $f = 0.1 + g = .0946$ nearly, and $x = 2.0946$ nearly.

4. This operation may be continued to any length, as by ſuppoſing $g = -.0054 + h$, and ſo on, and the value of $x = 2.09455147$ nearly.

By the firſt operation a nearer value of x may be found thus; ſince $f = .1$ nearly, and
$-1 + 10f + 6f^2 + f^3 = 0$,
$f = \frac{1}{10 + 6f + f^2}$, that is $f = \frac{1}{10 + .6 + .01} = .094$ true to the laſt figure, and $x = 2.094$.

In the ſame manner may the root of a pure equation be found, and this gives an eaſy method of approximating to the roots of numbers, which are not perfect powers.

This rule is applicable to numeral equations of every order, and, by aſſuming a general equation, general rules may be deduced for approximating to the roots of any propoſed equation. By a ſimilar method we may approximate to the roots of literal equations, which will be expreſſed by infinite ſeries.

ELEMENTS

OF

ALGEBRA.

PART III.

Of the Application of Algebra to Geometry.

CHAP. I.

General Principles.

GEOMETRY treats both of the magnitude and poſition of extenſion, and their connections.

Algebra treats only of magnitude. Therefore, of the relations which ſubſiſt in geometrical figures, thoſe of magnitude only can be immediately expreſſed by algebra.

The oppoſite poſition of ſtraight lines may indeed be expreſſed ſimply by the ſigns + and −. But, in order to expreſs the various other poſitions of geometrical figures by algebra, from the principles of geometry, ſome relations of magnitude muſt be found, which depend upon theſe poſitions, and which can be exhibited by equations: And converſely, by the ſame principles may the poſitions of figures be inferred from the equations denoting ſuch relations of their parts.

Though this application of algebra appears to be indirect, yet ſuch is the ſimplicity of the operations, and the general nature of its theorems, that inveſtigations, eſpecially in the higher parts of geometry, are generally eaſier and more expeditious by the algebraical method, though leſs elegant than by what is purely geometrical. The connections alſo, and analogies of the two ſciences eſtabliſhed by this application, have given riſe to many curious ſpeculations; geometry has been rendered far more extenſive and uſeful, and algebra itſelf has received conſiderable improvements.

I. *Of the Algebraical Expreſſion of Geometrical Magnitudes.*

A *line*, whether known or unknown, is repreſented by a ſingle letter; a *rectangle* is properly expreſſed by the product of the two letters repreſenting its ſides; and a *rectangular parallelopiped* by the product of three letters, two of which repreſent the ſides of any of its rectangular baſes, and the third the altitude.

Theſe are the moſt ſimple expreſſions of geometrical magnitudes, and any other which has a known proportion to theſe, may, in like manner, be expreſſed algebraically. Converſely, the geometrical magnitudes, repreſented by ſuch algebraical quantities, may be found, only the algebraical dimenſions above the third, not having any correſponding geometrical dimenſions, muſt be expreſſed by proportionals *.

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* All algebraical dimenſions above the third muſt be expreſſed by inferior geometrical dimenſions; and, tho' any algebraical quantities, of two and three dimenſions, may be immediately expreſſed by ſurfaces and ſolids reſpectively, yet it is generally neceſſary to expreſs them, and all ſuperior dimenſions, by lines.

The oppoſite poſition of ſtraight lines, it has been remarked, may be expreſſed by the ſigns + and —.

Thus,

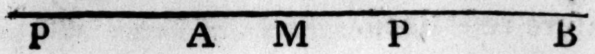
If, in any geometrical inveſtigation by algebra, each line is expreſſed by a ſingle letter, and each ſurface or ſolid by an algebraical quantity of two or three dimenſions reſpectively, then whatever legitimate operations are performed with regard to them, the terms in any equation derived will, when properly reduced, be all of the ſame dimenſion; and any ſuch equation may be eaſily expreſſed geometrically by means of proportionals, as in the following example.

Thus, if the algebraical equation $a^4+b^4=c^4-d^4$, is to be expreſſed geometrically, a, b, c, and d, being ſuppoſed to repreſent ſtraight lines; let $a:b:e:f:g$, in continued proportion, then $a^4:b^4::a:g$ and $a^4:a^4+b^4::a:a+g$; then let $a:c:h:k:l$, and $a^4:c^4::a:l$; alſo, let $c:d:m:n:p$, and $c^4:d^4::c:p$, or $c^4:c^4-d^4::c:c-p$. By combining the two former proportions, (Chap. 2. Part 1.) $c^4:a^4+b^4::l:a+g$, and combining the latter with this laſt found, $c^4-d^4:a^4+b^4::\overline{c-p}\times l::c\times\overline{a+g}$; therefore $\overline{c-p}\times l=c\times\overline{a+g}$, and $c:c-p::l:a+g$.

If any known line is aſſumed as 1, as its powers do not appear, the terms of an equation, including any of them, may be of very different dimenſions; and before it can be properly expreſſed by geometrical magnitudes, the deficient dimenſions muſt be ſupplied by powers of the 1. When an equation has been derived from geometrical relations, the line denoting 1 is known; and

when

Thus, let a point A be given in the line



AP, any ſegment AP taken to the right hand, being conſidered as poſitive, a ſegment Ap to the left is properly repreſented by a negative quantity. If a and b repreſent two lines; and if, upon the line AB from the point A, AP be taken towards the right equal to a, it may be expreſſed by $+a$; then PM taken to the left and equal to b, will be properly repreſented by $-b$, for AM is equal to $a-b$: If $a=b$, then M will fall upon A, and $a-b=0$: By the ſame notation, if b is greater than a, M will fall to the left of A; and in this caſe, if $2a=b$, and if Pp be taken equal to b, then $a-b=-a$ will repreſent Ap, which is equal to a, and ſituated to the left of A. This uſe of the ſigns, however, in particular

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when an aſſumed equation is to be expreſſed by the relations of geometrical magnitudes, the 1 is to be aſſumed.

In this manner may any ſingle power be expreſſed by a line. If it is x^5, then to 1, x find four quantities in continued proportion, ſo that $1 : x : m : n : p : q$, then $1 : q :: 1^5 : x^5$, or $q=x^5$, and ſo of others.

lar caſes, may be precluded, or in ſome meaſure reſtrained.

The poſitions of geometrical figures are ſo various, that it is impoſſible to give general rules for the algebraical expreſſion of them. The following are a few examples.

An angle is expreſſed by the ratio of its ſine to the radius ; a right angle in a triangle, by putting the ſquares of the two ſides equal to the ſquare of the hypothenuſe; the poſition of points is aſcertained by the perpendiculars from them on lines given in poſition ; the poſition of lines by the angles which they make with given lines, or by the perpendiculars on them from given points: The ſimilarity of triangles by the proportionality of their ſides which gives an equation, &c.

Theſe and other geometrical principles muſt be employed both in the demonſtration of theorems, and in the ſolution of problems. The geometrical propoſition muſt firſt be expreſſed in the algebraical manner, and the reſult after the operation muſt be expreſſed geometrically.

II. *The Demonſtration of Theorems.*

All propoſitions in which the proportions of magnitude only are employed, alſo all propoſitions expreſſing the relations of the ſegments of a ſtraight line, of their ſquares, rectangles, cubes, and parallelopipeds, are demonſtrated algebraically with great eaſe: Such demonſtrations, indeed, may in general be conſidered as an abridged notation of what are purely geometrical.

This is particularly the caſe in thoſe propoſitions, which may be geometrically deduced without any conſtruction of the ſquares, rectangles, &c. to which they refer. From the Firſt Propoſition of the Second Book of Euclid, the nine following may be eaſily derived in this manner, and they may be conſidered as proper examples of this moſt obvious application of algebra to geometry.

If certain poſitions are either ſuppoſed or to be inferred in a theorem, we muſt find, according to the preceding obſervations, the connection between theſe poſitions and ſuch relations of magnitude as can be expreſſed and reaſoned upon by algebra. The algebraical demonſtrations of the 12th and 13th propoſitions of the 2d Book of Euclid, require only the 47th of the I. El. The 35th and 36th of the 3d Book require only the 3. III. El. and 47. I. El.

From few ſimple geometrical principles alone, a number of concluſions, with regard to figures, may be deduced by algebra; and to this, in a great meaſure, is owing the extenſive uſe of this ſcience in geometry. If other more remote geometrical principles are occaſionally introduced, the algebraical calculations may be much abridged. The ſame is to be obſerved in the ſolution of problems; but ſuch in general are leſs obvious, and more properly belong to the ſtrict geometrical method.

III. *Of the Solution of Problems.*

Upon the ſame principles are geometrical problems to be reſolved. The problem is ſuppoſed to be conſtructed, and proper algebraical notations of the known and unknown magnitudes are to be ſought for, by means of which their connections may be expreſſed by equations. It may firſt be remarked, as was done in the caſe of theorems, that in thoſe problems which relate to the diviſion of the line, and the proportions of its parts, the expreſſion of the quantities, and the ſtating their relations by equations, are ſo eaſy as not to require any particular directions. But, when various poſitions of geometrical figures, and their properties are introduced, the ſolution requires more attention and ſkill. No general rules can be given on this ſubject, but the following obſervations may be of uſe.

I. The conſtruction of the problem being ſuppoſed, it is often farther neceſſary to produce ſome of the lines till they meet; to draw new lines joining remarkable points; to draw lines from ſuch points perpendicular or parallel to other lines, and ſuch other operations as ſeem conducive to the finding of equations; and for this purpoſe, thoſe eſpecially are to be employed, which divide the ſcheme into triangles that are given, right angled or ſimilar.

2. It is often convenient to denote by letters, not the quantities particularly ſought, but ſome others from which they can eaſily be deduced. The ſame may be obſerved of given quantities.

3. The proper notation being made, the neceſſary equations are to be derived by the uſe of the moſt ſimple geometrical principles, ſuch as the addition and ſubtraction of lines or of ſquares, the proportionality of lines, particularly of the ſides of ſimilar triangles, &c.

4. There muſt be as many independent equations as there are unknown quantities aſſumed in the inveſtigation, and from theſe a final equation may be inferred by the rules of Part I.

If the final equation from the problem be reſolved, the roots may often be exhibited geometrically; but the geometrical conſtruction of problems may be effected alſo, without reſolving the equation, and even without deducing a final equation, by the methods afterwards to be explained.

If the final equation is ſimple or quadratic, the roots being obtained by the common rules, may be geometrically exhibited by the finding of proportionals, and the addition or ſubtraction of ſquares.

By inſerting numbers for the known quantities, a numeral expreſſion of the quantities ſought will be obtained by reſolving the equation. But, in order to determine ſome particulars of the problem, beſides finding the unknown quantities of the equation, it may be farther neceſſary to make a ſimple conſtruction; or, if it is required that every thing be expreſſed in numbers, to ſubſtitute a new calculation in place of that conſtruction.

P R O P. I.

To divide a given ſtraight line AB *into two parts, ſo that the rectangle contained by the whole line, and one of the parts, may be equal to the ſquare of the other part.*

This is Prop. 11th II. B. of Euclid.

Let C be the point of diviſion, and let AB$=a$, AC$=x$, and then CB$=a-x$. From the problem $a^2-ax=x^2$; and this equation being reſolved (Chap. V. P. II.) gives $x=\pm\sqrt{a^2+\frac{a^2}{4}}-\frac{a}{2}$.

The quantity $\sqrt{a^2+\frac{a^2}{4}}$, is the hypothenuſe of a right angled triangle of which the two ſides are a, and $\frac{a}{2}$, and is therefore eaſily found; $\frac{a}{2}$ being taken from this line, gives $x=$AC, which is the proper ſolution. But if a line Ac be taken on the oppoſite ſide of A, and equal to the above mentioned hypothenuſe, together with $\frac{a}{2}$, it will repreſent the negative root $-\sqrt{a^2+\frac{a^2}{4}}-\frac{a}{2}$, and will give another ſolution; for in this caſe alſo $AB \times Bc = Ac^2$. But c is without the line AB, and therefore, if it is not conſidered as making a diviſion of AB, this negative root is rejected.

This ſolution coincides with what is given by Euclid. For $\sqrt{a^2+\frac{a^2}{4}}$ is equal (ſee the fig. of Prop. 11th II. B. Eucl. Simſon's edit.) to EB or EF, and therefore $x = \sqrt{a^2+\frac{a^2}{4}}-\frac{a}{2} = EF - EA = AF = AH$; and the point H correſponds to C in the preceding figure.

Beſides, if on $EF + EA = CF$ (inſtead of $EF - EA = FA$) a ſquare be deſcribed on the oppoſite ſide of CF from AG, BA produced will meet a ſide of it in a point, which if it be called K, will give $KB \times BA = KA^2$. K correſponds to c; and this ſolution will correſpond with the algebraical ſolution, by means of the negative root.

If CB had been called x, and AC $=a-x$, the equation would be $ax=a^2-2ax+x^2$, which gives $x=\frac{3a\pm\sqrt{5a^2}}{2}$, in which both roots are poſitive, and the ſolutions derived from them coincide with the preceding. If the ſolution be confined to a point within the line, then one of the poſitive roots muſt be rejected, for one of the roots of the compound ſquare from which it is derived is $x-\frac{3a}{2}$, a negative quantity, which in this ſtrict hypotheſis is not admitted. In ſuch a problem, however, both conſtructions are generally received, and conſidered even as neceſſary to a complete ſolution of it.

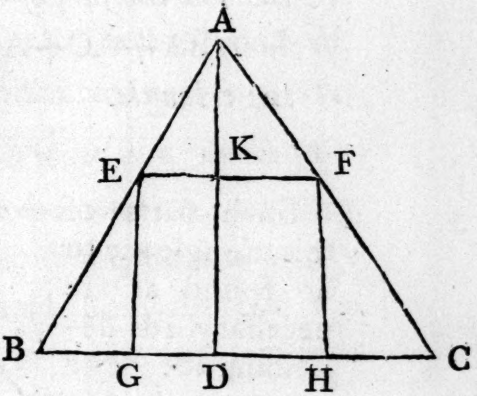
If a ſolution in numbers be required, let AB $=10$, then $x=\pm\sqrt{125}-5$. It is plain, whatever be the value of AB, the roots of this equation are incommenſurate, though they may be found, by approximation, to any degree of exactneſs required. In this caſe, $x=\pm 11.1803-5$, nearly, that is AC $=6.1803$ nearly; and A$c=16.1803$ nearly.

PROB.

PROB. II.

In a given Triangle ABC *to inſcribe a Square.*

Suppoſe it to be done, and let it be EFHG; from A let AD be perpendicular on the baſe BC, meeting EF in K.



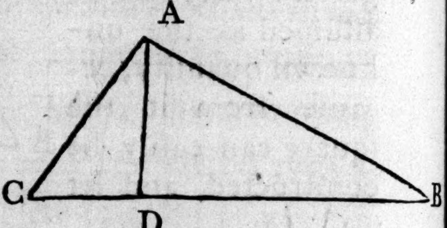
Let BC$=a$, and AD $=p$, both of which are given becauſe the triangle is given: Let AK be aſſumed as the unknown quantity, becauſe from it the ſquare can eaſily be conſtructed, and let it be called x: Then (KD$=$EG$=$)EF$=p-x$.

On account of the parallels EF, BC, AD : BC :: AK : EF; that is, $p : a :: x : p-x$, and $p^2-px=ax$, which equation being reſolved, gives $x=\frac{p^2}{p+a}$.

Therefore x or AK is a third proportional to $p+a$, and p, and may be found by 11. VI. El. The point K being found, the conſtruction of the ſquare is ſufficiently obvious.

PROB. III.

In the right angled Triangle ABC, *the baſe* BC *and the ſum of the perpendicular and ſides*, BA+AC+AD, *being given, to find the triangle.*



Such parts of this triangle are to be found as are neceſſary for deſcribing it: The perpendicular AD will be ſufficient for this purpoſe, and let it be called x: Let AB + AC + AD = a, BC = b, therefore BA+AC = $a-x$: Let BA—AC be denoted by y, then BA $=\frac{a+y-x}{2}$, and AC $=\frac{a-x-y}{2}$. But (47. I. El.) $BC^2=BA^2+AC^2$, which being expreſſed algebraically, becomes $b^2=\overline{\frac{a+y-x}{2}}\Big|^2+\overline{\frac{a-x-y}{2}}\Big|^2=\frac{a^2-2ax+x^2+y^2}{2}$. Likewiſe, from a known property of right angled triangles, BC×AD=BA×AC; that is $bx = \left(\frac{a+y-x}{2} \times \frac{a-y-x}{2} = \right) \frac{a^2-2ax+x^2-y^2}{4}$. This laſt equation being multiplied by 2, and added to the former, gives $b^2+2bx = a^2-2ax+x^2$, which being reſolved according to the rules of Part I. Chap. 5. gives $x = a+b-\sqrt{2ab+2b^2}$.

To conſtruct this: $a+b$ is the ſum of the perimeter and perpendicular, and is given; $\sqrt{2ab+2b^2} = \sqrt{\overline{a+b} \times 2b}$ is a mean proportional between $a+b$ and $2b$, and may be found; therefore, from the ſum of the perimeter and perpendicular, ſubtract the mean proportional between the ſaid ſum and double the baſe, and the remainder will be the perpendicular required.

From the baſe and perpendicular, the right-angled triangle is eaſily conſtructed.

In numbers, let $BA+AC+AD = 18.8 = a$; $BC = 10 = b$; then $AD = a+b-\sqrt{2ab+2b^2} = 28.8-\sqrt{576} = 4.8 = x$, and $BA+AC = 14$. By either of the firſt equations, $y^2 = 2b^2+2ax-a^2-x^2 = 4$ and $y = BA-AC = 2$; therefore $BA = 8$, and $AC = 6$.

The geometrical expreſſion of the roots of final equations ariſing from problems, may be found without reſolving them, by the interſection of geometrical lines. Thus, the roots of a quadratic are found by the interſections of the circle and ſtraight line, thoſe of a cubic and biquadratic, by the interſection of two conic ſections, &c.

The ſolution of problems may be effected alſo by the interſections of the *loci* of two intermediate equations without deducing a final equation: But theſe two laſt methods can only be underſtood by the doctrine of the *loci* of equations.

C H A P. II.

Of the Definition of Lines by Equations.

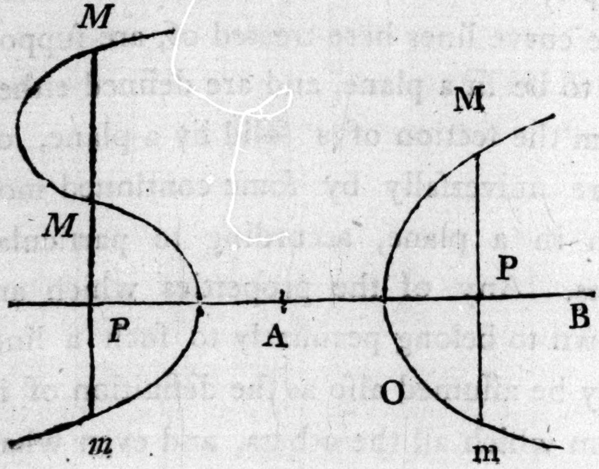
LINES which can be mathematically treated of, muſt be produced according to an uniform rule, which determines the poſition of every point of them.

This rule conſtitutes the *definition* of any line from which all its other properties are to be derived.

A ſtraight line has been conſidered as ſo ſimple, as to be incapable of definition. The curve lines here treated of, are ſuppoſed to be in a plane, and are defined either from the ſection of a ſolid by a plane, or more univerſally by ſome continued motion in a plane, according to particular rules. Any of the properties which are ſhewn to belong peculiarly to ſuch a line, may be aſſumed alſo as the definition of it, from which all the others, and even what, upon other occaſions may have been conſidered as the primary definition, may be demonſtrated. Hence lines may be defined in various methods, of which the moſt convenient is to be determined by the purpoſe in view. The ſimplicity of a definition, and the eaſe with which the other properties can be derived from it, generally give a preference.

DEFINITIONS.

I. When curve lines are defined by equations, they are ſuppoſed to be produced by the extremity of one ſtraight line, as PM moving in a given angle along another ſtraight line AB given in poſition, which is called the baſe.



II. The ſtraight line PM moving along the other, is called an *Ordinate*, and is uſually denoted by y.

III. The ſegment of the baſe AP between a given point in it A, and an ordinate PM, is called an *Abſciſs* with reſpect to that ordinate, and is denoted by x. The ordinate and abſciſs together are called *Co-ordinates*.

IV. If the relation of the variable abſciſs and ordinate, AP and PM, be expreſſed by an equation, which beſides x and y contains only known quantities, the curve MO deſcribed by the extremity of the ordinate, moving along the baſe, is called the *Locus* of that equation.

V. If the equation is finite, the curve is called *Algebraical* *. It is this claſs only which is here conſidered.

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* The terms *Geometrical* and *Algebraical*, as applied to curve lines, are uſed in different ſenſes, by different writers; there are ſeveral other claſſes of curves beſides what is here called algebraical, which can be treated of mathematically, and even by means of algebra. See Scholium at the end.

VI. The *dimenſions* of ſuch equations are eſtimated from the higheſt ſum of the exponents of x and y in any term.—According to this definition, the terms x^4, x^3y, x^2y^2, xy^3, y^4, are all of the ſame dimenſion.

VII. Curve lines are divided into *orders* from the dimenſions of their equations, when freed from fractions and ſurds.

In theſe general definitions, the ſtraight line is ſuppoſed to be comprehended, as it is the locus of ſimple equations. The loci of quadratic equations are ſhewn to be the conic ſections, which are hence called lines of the ſecond order, &c.

It is ſufficiently plain from the nature of an equation, containing two variable quantities, that it muſt determine the poſition of every point of the curve, defined by it in the manner now deſcribed: for if any particular known value of one of the variable quantities, as of x, be aſſumed, the equation will then have one unknown quantity only, and being reſolved, will give a preciſe number of correſponding values of y, which determine ſo many points of the curve.

As every point of the locus of an equation has the ſame general property, it muſt be one curve only, and from this equation all its properties may be derived. It is plain alſo, that any curve line defined from the motion of a point, according to a fixed rule, muſt either return into itſelf, or be extended *ad infinitum* with a continued curvature.

The equation, however, is ſuppoſed to be irreducible; becauſe, if it is not, the locus will be a combination of inferior lines; but this combination will poſſeſs the general properties of the lines of the order of the given equation.

It is to be obſerved all along, that the poſitive values of the ordinate, as PM, being taken upwards, the negative Pm will be placed downwards, on the oppoſite ſide of the baſe: And if poſitive values of the abſciſs, as AP, be aſſumed to the right from its beginning, the negative values A*P* will be upon the left, and from theſe the points of the curve M, m, on that ſide are to be determined.

In the general definition of curves it is uſual to ſuppoſe the co-ordinates to be at right angles. If the locus of any equation be deſcribed, and if the abſciſs be aſſumed on another baſe, and the ordinate be placed at a different angle, the new equation expreſſing their relation, though of a different form, will be of the ſame order as the original equation; and likewiſe will have, in common with it, thoſe properties which diſtinguiſh the equations of that particular curve.

This method of defining curves by equations may not be the fitteſt for a full inveſtigation of the properties of a particular curve; but, as their number is without limit, ſuch a minute inquiry concerning all, would be not only uſeleſs, but impoſſible. It has this great advantage, however, that many of the general affections of all curves, and of the diſtinct orders, and alſo ſome of the moſt uſeful properties of particular curves, may be eaſily derived from it.

I. *The Determination of the Figure of a Curve from its Equation.*

The general figure of the curve may be found by ſubſtituting ſucceſſively particular values of x the abſciſs, and finding by the reſolution of theſe equations the correſponding values of y the ordinate, and of conſequence ſo many points of the curve. If numeral values be ſubſtituted for x, and alſo certain numbers for the known letters, the reſolution of the equation gives numeral expreſſions of the ordinates ; and from theſe, by means of ſcales, a mechanical deſcription of the curve will be obtained, which may often be uſeful, both in pointing out the general diſpoſition of the figure, and alſo in the practical applications of geometry.

Some more general ſuppoſitions may be of uſe in determining the figure ; but theſe can be ſuggeſted only from the particular form of the equation in view. By ſuppoſing x to have certain relations to the known quantities, the values of y may become more ſimple, and the equation may be reduced to ſuch a form as to ſhew the direction of the curve, and ſome of its obvious properties.

The following general obſervations may alſo be laid down.

1. If in any caſe a value of y vaniſhes, then the curve meets the baſe in a point determined by the correſponding value of x. Hence by putting $y=0$, the roots of the equation, which in that ſituation are values of x, will give the diſtances on the baſe from the point aſſumed as the beginning of x, at which the curve meets it.

2. If at a particular value of x, y becomes infinite, the curve has an infinite arc, and the ordinate at that point becomes an aſymptote.

3. If, when x becomes infinitely great, y vaniſhes, the baſe becomes an aſymptote.

4. If any value of y becomes impoſſible, then ſo many interſections of the ordinate and curve vaniſh. If at any value of x all the values of y become impoſſible, the ordinate does not there meet the curve.

5. If two values of y become equal, and have the ſame ſign, the ordinate in that ſituation either touches the curve, or paſſes through an interſection of two of its branches, which is called a *punctum duplex*, or through an oval become infinitely little, called a *punctum conjugatum*.

In like manner is a *punctum triplex*, &c. to be determined.

The following example will illuſtrate this doctrine.

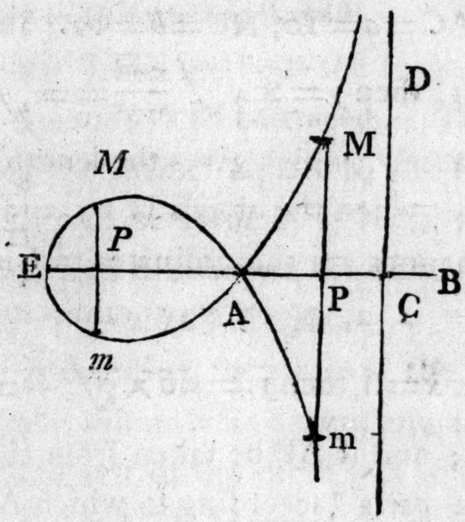
Let the equation be $ay^2 - xy^2 = x^3 + bx^2$:
Therefore, $y^2 = \frac{x^3+bx^2}{a-x}$ and $y = \pm\sqrt{\frac{x^3+bx^2}{a-x}}$ $= \pm\sqrt{\frac{a+b}{a-x}} \times x$.

Let AB be aſſumed as a baſe on which the abſciſſes are to be taken from A, and the ordinates perpendicular to it.

Since the two values of y are equal, but have oppoſite ſigns ; PM, and Pm which repreſent them, muſt be taken equal to each other on oppoſite ſides of AB ; and it is plain that the parts of the curve on the two ſides of AB, muſt be every way ſimilar and equal.

If x is made equal to a, then $y=x\sqrt{\frac{x+b}{0}}$ which is an algebraical expreſſion for infinity ; therefore, if AC is taken equal to a, the perpendicular CD will become an aſſymptote to the curve, which will have two infinite arcs (Obſ. 2.). If x is greater than a, the quantity under the radical ſign becomes negative, and the values of y are impoſſible ; that is, no part of the curve lies beyond CD (4.).

Both branches of the curve paſs through A, ſince $y=0$, when $x=0$ (1.). Let x be negative, and $y=\pm x\sqrt{\frac{b-x}{a+x}}$; the values of y will be poſſible, if x is not greater than b ; but, if $x=b$, then $y=0$, and if x is greater than b, the values of y become impoſſible ; that is, if the abſciſs AP be taken to the left of A, and leſs than b, there will be two real equal values of y, viz. *PM*, *Pm* on the oppoſite ſides as before; if AE be taken equal to b the curve will paſs through E, and no part of it is beyond E (1. and 4.).



The portion between A and E is called a *Nodus*.

If y be put $=0$, then the values of x are $0, 0, -b$. That is, the curve paſſes twice through A, or A is a punctum duplex, and it paſſes alſo through E as before (1.).

The mechanical deſcription of curves, mentioned in the beginning of this ſection, may be illuſtrated by the preceding example. For this purpoſe, let any numeral values of a and b be aſſumed ; and if ſucceſſive numeral values of x be inſerted, correſponding numeral values of y will be obtained, by which ſo many points in the curve may be conſtructed.

Let AC$=a=10$; AE$=b=12$; and, firſt let $x=1$, then $y=\pm x\sqrt{\frac{x+b}{a-x}}=\pm\sqrt{\frac{13}{3}}=$ ±1.2 nearly, which gives the length of the ordinates when the abſciſs is 1 ; and in the ſame manner are the ordinates to be found when x is 2, 3, or any other number. Thus, if $x=6$, then $y=\pm6\times\sqrt{\frac{18}{2}}=12.73$ nearly ; and if AP be taken from the ſcale of equal parts (according to which AB and AE are ſuppoſed to be laid down) and equal to 6, then PM, Pm, being taken from the ſame ſcale, each equal to 12.73, will give the points of the curve M, m. In like manner, if $x=-9$, $y=\pm9\sqrt{\frac{3}{19}}=\pm$ 3.58, nearly, and if A$P=9$, then *PM*, *Pm* being taken from the ſame ſcale equal to 3.58, will give the points *M*, *m*. In the ſame manner may any number of points be found, and theſe being joined, will give a repreſentation of the curve, which will be more or leſs juſt, according to the number of points found, and the accuracy of the ſeveral operations employed.

By the ſame methods the locus of any other equation is to be traced: Thus, by varying the former equation, the figure of its locus will be varied. If $b=0$, then the points A and E coincide, the nodus vaniſhes, and A is called a *Cuſpis*.

If b is negative, then E is to the right of A, which will now be a punctum conjugatum. The reſt of the curve will be between E and C, and CD becomes an aſymptote.

If $a=0$, then $-xy^2=x^3-bx^2$, or $y^2=bx-x^2$, which is an equation to the circle of which $b=AE$ is the diameter.

II. *General Properties of Curves from their Equations.*

The general properties of equations lead to the general affections of curve lines. For example, A ſtraight line may meet a curve in as many points as there are units in the dimenſion of its equation; for ſo many roots may that equation have. An aſymptote may cut a curve line in as many points excepting two as it has dimenſions, and no more. The ſame may be obſerved of the tangent.

Impoſſible roots enter an equation by pairs, therefore the interſection of the ordinate and curve muſt vaniſh by pairs.

The curves of which the number expreſſing the order is odd, muſt have at leaſt two infinite arcs; for the abſciſs may be ſo aſſumed, that, for every value of it, either poſitive or negative, there muſt be at leaſt one value of y, &c.

The properties of the coefficients of the terms of equations mentioned Part II. Ch. 1. furniſh a great number of the curious and univerſal properties of curve lines. For example, the ſecond term of an equation is the ſum of the roots with the ſigns changed, and if the ſecond term is wanting, the poſitive and negative roots muſt be equal. From this it is eaſy to demonſtrate, " That, if each of two parallel ſtraight lines meet a curve line in as many points as it has dimenſions, and if a ſtraight line cut theſe two parallels, ſo that the ſum of the ſegments of each on one ſide be equal to the ſum of the ſegments on the other, this ſtraight line will cut any other line parallel to theſe in the ſame manner." Analagous properties, with many other conſequences from them, may be deduced from the compoſition of the coefficients of the other terms.

Many properties of a particular order of curves may be inferred from the properties of equations of that order. Thus, " if a ſtraight line cut a curve of the third order in three points, and if another ſtraight line be drawn, making a given angle with the former, and cutting the curve alſo in three points, the parallelopiped by the ſegments of one of theſe lines between its interſection with the other, and the points where it meets the curve, will be to the parallelopiped by the like ſegments of the other line in a given ratio." This depends upon the compoſition of the abſolute term, and may be extended to curves of any order.

III. *The Subdiviſion of Curves.*

As lines are divided into orders from the dimenſions of their equations, in like manner, from the varieties of the equations of any order, may different *Genera* and *Species* of that order be diſtinguiſhed, and from the peculiar properties of theſe varieties, may the affections of the particular curves be diſcovered.

For this purpoſe a complete general equation is aſſumed of that order, and all the varieties in the terms and coefficients which can affect the figure of the locus are enumerated.

It was formerly obſerved, that the equations belonging to any one curve, may be of various forms, according to the poſition of the baſe, and the angle which the ordinate makes with it, though they be all of the ſame order, and have alſo certain properties, which diſtinguiſh them from the other equations of that order.

The locus of ſimple equations is a *ſtraight line*. There are *three* ſpecies of lines of the ſecond order, which are eaſily ſhewn to be the *conic ſections*, reckoning the circle and ellipſe to be one. *Seventy-eight* ſpecies have been numbered of the third order: And, as the ſuperior orders become too numerous to be particularly reckoned, it is uſual only to divide them into certain general claſſes.

A complete arrangement of the curves of any order, would furniſh canons, by which the ſpecies of a curve whoſe equation is of that order might be found.

IV. *Of the place of Curves defined from other principles, in the Algebraical Syſtem.*

If a curve line be defined from the ſection of a ſolid, or from any rule different from what has been here ſuppoſed, an equation to it may be derived, by which its order and ſpecies in the *algebraical* ſyſtem may be found. And, for this purpoſe, any baſe and any angle of the co-ordinates may be aſſumed, from which the equation may be moſt eaſily derived, or may be of the moſt ſimple form.

The three *Conic Sections* are of the ſecond order, as their equations are univerſally quadratic; the *Ciſſoid* of the ancients is of the third order, and the forty-ſecond ſpecies, according to Sir Iſaac Newton's enumeration; this is the curve defined by the equation in page 241, when $b=0$. The curve delineated in page 239 is the 41ſt ſpecies. When b is negative in that equation, the locus is the 43d ſpecies. The *Conchoid* of *Nicodemes* is of the fourth order; the *Caſſinian* curve is alſo of the fourth order, &c.

It is to be obſerved, that not only the firſt definition of a curve may be expreſſed by an equation, but likewiſe any of thoſe theorems called *loci*, in which ſome property is demonſtrated to belong to every point of the curve. The expreſſion of theſe propoſitions by equations is ſometimes difficult; no general rules can be given; and it muſt be left to the ſkill and experience of the learner.

SCHO-

SCHOLIUM.

This method of treating curve lines by equations, beſides the uſes already hinted at, has many others, which do not belong to this place ; ſuch are, the finding the tangents of curves, their curvature, their areas and lengths, &c. The ſolution of theſe problems has been accompliſhed by means of the equations to curves, though by employing, concerning them, a method of reaſoning different from what has been here explained.

CHAP. III.

I. *Conſtruction of the Loci of Equations.*

THE deſcription of a curve, according to the definition of it, is aſſumed in geometry as a *Poſtulate* *.

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* A poſtulate in geometry ſeems to be improperly called a mechanical principle. No geometrical line whatever, not even the ſtraight line or circle, can be deſcribed mechanically according to the mathematical definition; and therefore the ſolutions of problems by the conic ſections, or by any of the higher orders, is to be conſidered, in theory, as equally perfect with thoſe by the circle and ſtraight lines. It is a rule in ſtrict geometry, not to employ a curve line in the ſolution of a problem, if it can be performed by means of a line of an inferior order; but, when a practical ſolution is required, then thoſe lines, of whatever order, or of whatever claſs, and thoſe methods of deſcribing

If the properties of a particular curve are inveſtigated, it will appear that it may be deſcribed from a variety of data different from thoſe aſſumed in the poſtulate, by demonſtrating the dependence of the former upon the latter.

As the definitions of a curve may be various, ſo alſo may be the poſtulates, and a definition is frequently choſen from the mode of deſcription connected with it. The particular object in view, it was formerly remarked, muſt determine the proper choice of

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ſcribing them are to be preferred, by which the conſtruction required may be moſt eaſily and accurately performed. Thus, even the 2d and 3d Prop. of I. B. of Euclid are conſtructed in practice, with much more eaſe and accuracy, by transferring a diſtance in a pair of compaſſes, than by the methods there deſcribed; but that principle not being aſſumed by Euclid as a poſtulate, could not be admitted in the conſtruction of any problem in his elements. There are but few mechanical operations which admit of tolerable accuracy, and hence the great advantage of arithmetical calculations in the practical arts founded on geometry. By theſe the more complicated conſtructions of geometry are reduced to thoſe ſimple operations which are found by experience to be capable of greateſt exactneſs.

of a definition; the ſimplicity of it, the eaſe with which the other properties of the figure may be derived from it, and ſometimes even the eaſe with which it can be executed mechanically, may be conſidered as important circumſtances.

In the ſtraight line, the circle, the conic ſections, and a few curves of the higher orders, the moſt convenient definitions, and the poſtulates connected with them, are generally known and received. An equation to a curve may alſo be aſſumed as a definition of it, and the deſcription of it, according to that definition, may be conſidered as a poſtulate; but, if the geometrical conſtruction of problems is to be inveſtigated by means of Algebra, it is often uſeful to deduce from the equation to a curve thoſe data, which, from the geometrical theory of the curve, are known to be neceſſary to its deſcription in the original poſtulate, or in any problem founded upon it. This is called *conſtructing the locus* of an equation, and from this method are generally derived the moſt elegant conſtructions which can be obtained by the uſe of algebra. In the following ſection, there is an example of a problem reſolved by ſuch conſtructions.

Sometimes a mechanical deſcription of a curve line defined by an equation is uſeful; and, as the exhibition of it by ſuch a motion as is ſuppoſed in that definition, is rarely practicable, it generally becomes neceſſary to contrive ſome more ſimple motion which may in effect correſpond with the other, and may deſcribe the curve with the degree of accuracy which is wanted. Frequently, indeed, the only method which can be conveniently practiſed, is the finding a number of points in the curve by the reſolution of numeral equations, in the manner mentioned in Sect. 1. of this Chapter, and then joining theſe points by the hand; and, though this operation is manifeſtly imperfect, it is on ſome occaſions uſeful.

II. *Solution of Problems.*

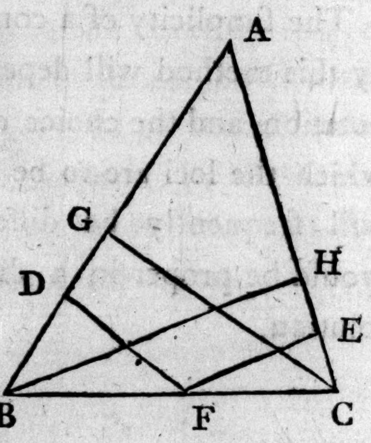
The ſolution of geometrical problems by algebra is much promoted, by deſcribing the loci of the equations ariſing from theſe problems.

For this purpoſe, equations are to be derived according to the methods formerly deſcribed, and then to be reduced to two, containing each the ſame two unknown quantities. The loci of theſe equations are to be deſcribed, the two unknown quantities being conſidered as the co-ordinates, and placed at the ſame angle in both. The co-ordinates at an interſection of the loci will be common to both, and give a ſolution of the problem.

The ſimplicity of a conſtruction obtained by this method will depend upon a proper notation, and the choice of the equations of which the loci are to be deſcribed. Theſe will frequently be different from what would be proper in a different method of ſolution.

PROB. IV.

To find a point F *in the baſe of the given triangle* ABC, *ſo that the ſum of the ſquares of* FE, FD *drawn from it perpendicular upon the two ſides, may be equal to a given ſpace.*



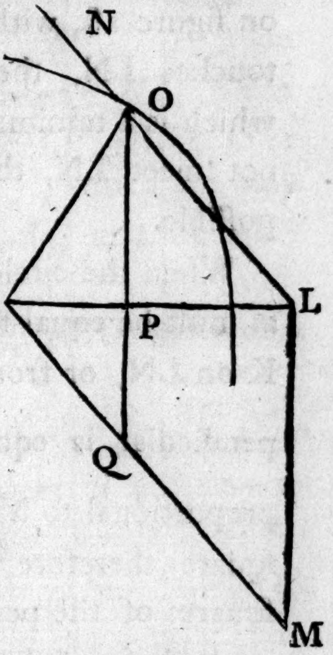
Draw BA, CG perpendicular on the two ſides, and let FD$=x$, FE$=y$, BF$=z$, BC$=b$, BH$=p$, CG$=r$, and the given ſpace FD$^2+$FE$^2=m^2$.

From ſimilar triangles, $z:x::b:r$, and $z=\frac{bx}{r}$.

Alſo $b-z:y::b:p$, and $z=b-\frac{by}{p}$; therefore $\frac{bx}{r}=b-\frac{yb}{p}$; that is, $y=p-\frac{px}{r}$, an equation to a ſtraight line.

But $x^2+y^2=m^2$, of which the locus is a circle, having m for the radius. By conſtructing theſe loci, their interſection will give a ſolution of the problem.

Let KL$=$CG ($=r$) be at right angles to LM$=$BH ($=p$), join KM, to which let LN be parallel; LN is the locus of the equation $y=p-\frac{px}{r}$; for, let any line OPQ be drawn parallel to LM, if KP$=x$, then PQ$=\frac{px}{r}$, and QO$=$LM$=p$, therefore PO$=y=p-\frac{px}{r}$.



About the center K, with a diſtance equal to the line m, let a circle be deſcribed; that circle will be the locus of the equation $m^2=x^2+y^2$, for it is plain that if OP be any perpendicular from the circumference upon KL, KP being x, OP will be y. Either of the points therefore in which theſe two loci interſect each other, as O, will give OP an ordinate in both equations,

tions, KP being the common abſciſs ; therefore KP, OP are the two perpendiculars required, from which the point F is eaſily found.

The conſtruction might have been made on figure 1ſt, with fewer lines. If the circle touches LN, there is only one ſolution which is a minimum ; and if the circle does not meet LN, the problem becomes impoſſible.

When the circle touches LN, the radius m muſt be equal to the perpendicular from K on LN, or from L on KM. This perpendicular is equal to $\frac{pr}{\sqrt{p^2+r^2}}$, or a fourth proportional to MK, KL, and LM, and its ſquare therefore is the leaſt ſum of the ſquares of the perpendiculars from a point in the baſe on the two ſides.

It may be remarked alſo, that the point which gives the ſum of the ſquares a minimum, is found by dividing the baſe, in the proportion of the ſquares of the two ſides of the triangle ; and this is eaſily demonſtrated from the preceding conſtruction.

PROB.

PROB. V.

Between two given Lines to find two mean Proportionals.

Let the lines be a and b, and let the two means be x and y; therefore $a : x : y : b$, and hence $ay = x^2$, and $bx = y^2$, which are both equations to the parabola, and are eaſily conſtructed. The co-ordinates at the interſection of theſe two loci, will be the means required.

If one unknown quantity only is aſſumed, or if it is convenient to deduce a final equation containing only one, the conſtruction of the roots is to be obtained by the method mentioned in the next ſection.

SCHOLIUM.

The conſtructions of the two preceding problems are geometrical; but it is ſometimes convenient to have a practical ſolution, by the mechanical deſcription either of the algebraical lines employed in the geometrical ſolution, or of other geometrical lines by which it can be effected. But few of theſe are tolerably accurate, ſo that, in general, by means of calculation, the practical operations are all reduced to what may be performed by a rule and a compaſs.

III. *Conſtruction of Equations.*

The roots of an equation, containing only one unknown quantity, may be found by the interſection of lines, the product of whoſe dimenſions is equal to the dimenſion of that equation. And hence problems are reſolved without an algebraical ſolution of the equation ariſing from them.

Thus cubic and biquadratic equations may be conſtructed by the interſections of two conic ſections as the circle and parabola, which are generally aſſumed as being moſt eaſily deſcribed.

In order to find theſe conſtructions, a new equation is to be aſſumed, containing two variable quantities, one of which is the unknown quantity of the given equation, and the other by ſubſtitution is to be inſerted alſo in the given equation; the interſection of the loci of theſe equations will exhibit the roots required.

Canons may be deviſed for the conſtruction of particular orders, without aſſuming the new equation.

The final equation from Prob. 5. would be $x^3=a^2b$, which being conſtructed according to the rules, exhibits the common geometrical ſolution of that problem by the circle and parabola.

If an equation be aſſumed, as $ay=x^2$, the other, by ſubſtitution, becomes $xy=ab$; the locus of the former is a parabola, and of the latter an hyperbola, one of its aſſymptotes being the baſe, and the co-ordinates at their interſection will repreſent x and y; the firſt of the two means is x, and in this caſe y is the other.

Equations alſo might be aſſumed ſo as to give a ſolution of this problem by other combinations of two of the conic ſections, one of them not being the circle.

As geometrical magnitudes may be repreſented by algebra, ſo algebraical quantities and numbers may be repreſented by lines. Hence this conſtruction of equations has ſometimes been uſed as an eaſy method of approximation to the roots of numeral equations. For this purpoſe, the neceſſary ſtraight lines muſt be laid down by means of a ſcale of equal parts, and the curve lines, on whoſe interſection the conſtruction depends, muſt be actually deſcribed; the linear roots being meaſured on the ſcale, will give the numbers required. Theſe operations may be performed with ſufficient accuracy for certain purpoſes; but, as they depend on mechanical principles, the approximation obtained by them cannot be continued at pleaſure; and hence it is ſeldom uſed, except in finding the firſt ſtep of an approximation, which is to be carried on by other methods.

SCHOLIUM.

If the relation between the ordinate and abſciſs be fixed, but not expreſſible by a finite equation, the curve is called *Mechanical* * or *Tranſcendental.* This claſs is alſo ſometimes defined by equations, by ſuppoſing either x or y in a finite equation to be a curve line, of which the relation to a ſtraight line cannot be expreſſed in finite terms.

If the variable quantities x or y enter the exponents of any term of an equation, the locus of that equation is called an *Exponential Curve.*

Many properties of theſe two claſſes of curves may be diſcovered from their equations.

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* The term *Mechanical,* in this place, is uſed merely as the name of a particular claſs of curves, without implying that they have any more dependence on the principles of Mechanics or Phyſics, than the *algebraical* curves which have been treated of.

APPENDIX.

[illegible]

ELEMENTS

[illegible]

ALGEBRA

APPENDIX

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ELEMENTS

OF

ALGEBRA.

APPENDIX.

APPENDIX.

No. I.

Continued Fractions.

1. THE manner of reducing fractions to their leaſt terms has ſuggeſted another form, in which it is often convenient to have them expreſſed. The nature of this form, or of what is called a *continued fraction*, will be eaſily underſtood by an example.

Let there be given $\frac{314159}{100000}$, or, which is the ſame, $3+\frac{14159}{100000}$. Since $100000=7\times 14159+887$, therefore $\frac{14159}{100000}=\frac{14159}{7\times 14159+887}$ $=\frac{1}{7+\frac{887}{14159}}$, and $\frac{314159}{100000}=3+\frac{1}{7+\frac{887}{14159}}$.

Now, $\frac{887}{14159}=\frac{887}{15\times 887+854}=\cfrac{1}{15+\frac{854}{887}}$, and ſubſtituting this for $\frac{887}{15159}$, in the value of $\frac{314159}{100000}$ already found, we have $\frac{314159}{100000}=$

$$3+\cfrac{1}{7+\cfrac{1}{15+\frac{854}{887}}}.$$

Again, $\frac{854}{887}=\frac{854}{854+33}=\cfrac{1}{1+\frac{33}{854}}$, which being ſubſtituted as before, gives $\frac{314159}{100000}=$

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\frac{33}{854}}}}.$$

By continuing to reſolve the fraction $\frac{33}{854}$, and thoſe formed from it, in the ſame manner, we have $\frac{33}{854}=\cfrac{1}{25+\frac{29}{33}}$, and $\frac{29}{33}=\cfrac{1}{1+\frac{4}{29}}$, and $\frac{4}{29}=\cfrac{1}{7+\frac{1}{4}}$, whence, by ſubſtitution,

$$\frac{314159}{100000}=3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{25+\cfrac{1}{1+\cfrac{1}{7+\frac{1}{4}}}}}}}.$$

2. By the ſame means, may any fraction be reſolved into the form,

$$a+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\text{, \&c.}$$

and it is then called a *continued fraction.* This formula may be expreſſed in words, by ſaying, that the reciprocal of *d* is to be added to *c*, the reciprocal of that ſum to *b*, and the reciprocal of this laſt ſum to *a*, in order to have the fraction required.

3. The inverſe of the preceding operation, or that by which a continued fraction is reduced into the form of an ordinary one, is eaſily derived from this explanation. Thus, if the fraction be

$$a+\frac{1}{b}+\frac{1}{c}+\frac{1}{d},$$

then the reciprocal of *d* added to *c*, or $c+\frac{1}{d}=\frac{dc+1}{d}$, the reciprocal of which, $=\frac{d}{cd+1}$, being added to *b*, gives $b+\frac{d}{cd+1}=\frac{bcd+b+d}{cd+1}$; and the reciprocal of this laſt, viz. $\frac{cd+1}{bcd+b+d}$, being added to a, gives

$$\frac{abcd+ab+ad+cd+1}{bcd+b+d}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}.$$

4. One of the chief advantages derived from this manner of expreſſing fractions is, that it enables us to find a ſeries of other fractions that approach in value to any given one, and each of them expreſſed in the leaſt numbers poſſible.

Thus, in the given example $\frac{314159}{100000}$, which is known to expreſs nearly the proportion of the circumference of a circle to its diameter, and which we may call π, if we take only the two firſt terms of the continued fraction into which it was reſolved, viz. $3+\frac{1}{7}$, we have $\pi=\frac{22}{7}$ nearly; and this is the theorem of Archimedes.

If we take the three firſt terms, we have

$$\pi=3+\cfrac{1}{7+\cfrac{1}{15}}=3+\frac{15}{106}=\frac{333}{106},$$

which is nearer to the truth than the former.

And, if we take the four firſt terms, making

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113},$$

we have the proportion of Metius, more exact than either of the preceding. Theſe reſults are alternately greater and leſs than the truth.

5. Among continued fractions, thoſe have been peculiarly diſtinguiſhed in which the denominators, after a certain number of changes, are continually repeated in the ſame order. Such is the fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \text{, \&c.}}}}}$$

The amount of this fraction, though continued *ad infinitum*, may be eaſily found; for, if x be the amount of all the terms but the firſt, which is an integer, $(=1)$, ſo that

$$x = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \text{, \&c.}}}}}$$

Then, ſince, after the 2d, all the fractions return in the ſame order, their amount is alſo $= x$; therefore

$x=$

$$x=\frac{1}{2}+\frac{1}{3+x}$$

Hence $x=\frac{3+x}{6+2x+1}$, and $x^2+3x=\frac{3}{2}$, and $x=\frac{-3+\sqrt{15}}{2}$. Therefore $x+1$, or the whole ſeries, $=\frac{-1+\sqrt{15}}{2}$.

In general, if $x=\frac{1}{a}+\frac{1}{b}+\frac{1}{a}+$, &c. we find $x=-\frac{b}{2}\pm\sqrt{\frac{1}{4}b^2+\frac{b}{a}}$. Though the denominators did not return in the ſame order till after a greater interval, the value of the fraction would ſtill be expreſſed by the root of a quadratic equation; and converſely, the roots of all quadratic equations may be expreſſed by periodical continued fractions, and by that means may often be very readily approximated in numbers, without the extraction of the ſquare root.

6. The reduction of decimal, into the form of continued fractions, ſometimes renders the law of their continuation evident.

Thus, we know that $\sqrt{2}=1.4121356+$; but, from the bare inſpection of this decimal, we diſcover no rule for its farther continuation. If, however, it be reduced into a continued fraction, we find it

$$=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+,\ \&c.}}}$$

and we ſee in what manner it may be carried on to any degree of exactneſs.

APPENDIX.

No. II.

Indeterminate Problems.

THE indeterminate problems of the firſt and ſecond order, after all the unknown quantities, but two, have been exterminated, lead to equations of four different forms.

$$\text{I.}\quad y = \frac{a+bx}{c}.$$

$$\text{II.}\quad y = \frac{a}{b+cx}.$$

$$\text{III.}\quad y = \frac{a+bx}{c+dx}.$$

$$\text{IV.}\quad y = \sqrt{a+bx+cx^2}.$$

In all theſe formulas, *a*, *b*, *c*, *d*, denote given numbers. In the three firſt, x is to be found, ſo that y may be an integer; and, in the fourth, x is to be found, ſo that y may be a rational quantity.

1. When $y=\frac{a+bx}{c}$, the ſolution depends on the rule for deducing a fraction to its loweſt terms. See the text, page 125.

2. When $y=\frac{a}{b+cx}$, it is plain that, in order to have y an integer, $b+cx$ muſt be a diviſor of a. Let d be one of its diviſors, then, if $b+cx=d$, $x=\frac{d-b}{c}$; ſo that among the diviſors of a, we muſt find one, if poſſible, from which b being taken away, the remainder may be diviſible by c: the quotient is a value of x.

3. When $y=\frac{a+bx}{c+dx}$, if d be a diviſor of b, x will be taken out of the numerator, if we divide it by $dx+c$; and this form is then reduced to the preceding. But, if d is not a diviſor of b, multiply both ſides by d, then $dy=\frac{ad+bdx}{c+dx}$, or, dividing $bdx+ad$

by $dx+c$, $dy=b+\frac{ad-bc}{c+dx}$, and ſo x is found by making $c+dx$ equal to a diviſor of $ad-bc$.

Example.

Let $2xy+x+y=195$; then $y(2x+1)=195-x$, and $y=\frac{195-x}{1+2x}$. Therefore $2y=\frac{390-2x}{1+2x}=-1+\frac{391}{1+2x}$. Now $391=17\times 23$, and if $1+2x=17$, $x=8$, and $y=11$.

4. When $y=\sqrt{a+bx+cx^2}$, and x is to be found, ſo that y may be a rational quantity. Here there are four caſes, according to the nature of the coefficients a, b, and c.

1*mo*, If a be a ſquare number, as for inſtance g^2, ſo that the formula is $\sqrt{g^2+bx+cx^2}$. Suppoſe $\sqrt{g^2+bx+cx^2}=g+mx$, then $g^2+bx+cx^2=g^2+2mgx+m^2x^2$, or $bx+cx^2=2mgx+m^2x^2$, that is, $b+cx=2mg+m^2x$, and $x=\frac{2mg-b}{c-m^2}$.

If for x this value of it be ſubſtituted in the given formula, its irrationality will diſappear, and $\sqrt{g^2+bx+cx^2}=\frac{cg-bm+gm^2}{c-m^2}$; m may be aſſumed, therefore, equal to any quantity, poſitive or negative, integral or fractional, and the correſponding value of x will anſwer the conditions preſcribed.

2*do*, If c be a ſquare number, as g^2, then let $\sqrt{a+bx+g^2x^2}=m+gx$. Hence $a+bx+g^2x^2=m^2+2mgx+g^2x^2$, or $a+bx=m^2+2mgx$. Therefore $x=\frac{m^2-a}{b-2mg}$, and $\sqrt{a+bx+g^2x^2}=\frac{bm-gm^2-ag}{b-2mg}$. Here m, as before, may be aſſumed at pleaſure.

3*tio*, Though neither a nor c are ſquare numbers, yet if $a+bx+cx^2$ can be reſolved into two ſimple factors, as $f+gx$, and $h+kx$, the irrationality of the formula may be taken away. For, let $\sqrt{a+bx+cx^2}=\sqrt{(f+gx)(h+kx)}=m(f+gx)$, and $(f+gx)(h+kx)=m^2(f+gx)^2$, or $h+kx=m^2(f+gx)$, and $x=\frac{fm^2-h}{k-gm^2}$. By the ſubſtitution of this value of x, (m being aſſumed at pleaſure), the irrationality will be removed, as before.

4to, The fourth caſe, in which the formula $a+bx+cx^2$ may be rendered a complete ſquare, is when it can be divided into two parts, one of which is a complete ſquare, and the other a product of two ſimple factors. For $a+bx+cx^2$ is then of the form p^2-qr, p, q, and r, being quantities into which there enters no power of x higher than the firſt: And, if we aſſume $\sqrt{p^2+qr}=p+mq$, p^2 will be exterminated, and the remainder, being divided by q, will be a ſimple equation, from which x may be eaſily determined.

Theſe methods of removing the irrationality of the preceding formula are to be particularly attended to, as being of great uſe in the higher geometry.

med as poſitive), the imaginary y will be removed, as before.

4thly. The fourth caſe, in which the formula $x^4 + bx + [illegible]$ may be rendered a complete ſquare, in which it can be divided into two parts, one of which is a complete ſquare, and the other a product of two ſimple factors, [illegible] of the form [illegible], [illegible] and [illegible] being quantities into which there enters no power of x higher than the firſt; and, if we aſſume [illegible] will be determinated, and the remainder, being divided by [illegible], will give a ſimple equation, from which x may be eaſily determined.

The methods of [illegible] the [illegible] of (2) [illegible] to be particularly [illegible] the higher geometry.

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APPENDIX.

No. III.

Extermination of unknown Quantities, and of Radicals.

1. THE extermination of unknown quantities, from ſimple equations, may always be performed by ſubſtitutions, but ſometimes more eaſily by the addition, and ſubtraction of the equations themſelves. This laſt method will be beſt explained by an example.

Suppoſe that $ax+by=c$, and $dx+fy=g$, where a, b, c, &c. are given quantities, that may be either poſitive or negative. To exterminate y, multiply the firſt equation by f, the coefficient of y in the ſecond, and the ſecond by b, the coefficient of y in the firſt, and ſubtract the latter from the former; we have

$$afx+bfy=cf$$
$$bdx+bfy=bg,$$

And therefore $afx-bdx=cf-bg$,

Or $$x=\frac{cf-bg}{af-bd}.$$

In the ſame manner, $$y=\frac{ag-cd}{af-bd}.$$

2. Let there be now three unknown quantities, and three equations, viz.

$$ax+by+cz+d=0,$$
$$fx+gy+hz+l=0,$$
$$mx+ny+pz+q=0;$$

multiply the firſt by h, and the ſecond by c, then, by the ſubtraction of the one from the other, z will be exterminated; for

$$ahx+bhy+chz+dh=0.$$
$$cfx+cgy+chz+cl=0,$$

and $(ah-cf)x+(bh-cg)y+dh-cl=0$.

In the ſame manner, an equation without z, is found from the ſecond and third, viz.

$$(fp-hm)x+(gp-hn)y+lp-hq=0.$$

From the two laſt equations, y is to be exterminated, as in the former example, and we will then have,

$$x = \frac{(bh-cg)(lp-hq)-(gp-hn)(dh-cl)}{(ah-cf)(gp-hn)-(bh-cg)(fp-hm)}.$$

Alſo $y = \dfrac{(ah-cf)(lp-hq)-(fp-hm)(dh-cl)}{(ah-cf)(gp-hn)-(bh-cg)(fp-hm)}$;

And $z = \dfrac{(fn-gm)(dg-bl)-(ag-bf)(ln-gq)}{(ag-bf)(gp-hn)-(cg-bh)(fn-gm)}$.

3. By the ſame method may the unknown quantities be exterminated from equations of the higher orders; but it will then be convenient to write ſingle letters inſtead of the coefficients, whether known or unknown, of the different powers of the quantity to be exterminated. For example, let $y^2-3xy+ay-x^2=0$,

and $y^2-by+2ax-4x^2+b^2=0$;

to exterminate y.

As the coefficient of y^2 is 1, in both equations, no ſubſtitution for it is neceſſary, but that the method may be general, we ſhall ſuppoſe $A=1$, $B=a-3x$, $C=-x^2$; alſo, in the ſecond equation, $D=1$, $E=-b$,

$F=2ax-4x^2+b^2$; the two equations are then

$$Ay^2+By+C=0,$$

$$\text{and}\quad Dy^2+Ey+F=0.$$

To exterminate y^2, multiplying the firſt by D, and the ſecond by A, we have

$$ADy^2+BDy+CD=0,$$

$$ADy^2+AEy+AF=0;$$

Therefore $(BD-AE)y+CD-AF=0$,

$$\text{and}\quad y=\frac{AF-CD}{BD-AE}.$$

Again, to obtain another value of y, multiply the firſt equation by F, and the ſecond by C; then

$$AFy^2+BFy+CF=0,$$

$$\text{and}\quad CDy^2+CEy+CF=0,$$

therefore $(AF-CD)y^2+(BF-CE)y=0$.

that is, $(AF-CD)y+BF-CE=0$,

$$\text{or,}\ y=\frac{CE-BF}{AF-CD},$$

$$\text{therefore}\ \frac{AF-CD}{BD-AE}=\frac{CE-BF}{AF-CD},$$

or $(AF-CD)^2=(BD-AE)(CE-BF)$.

As *y* does not enter into this equation, if we reſtore the values of A, B, and C, we have an equation involving only *x* and known quantities, and have thus obtained a general theorem for the extermination of one of the unknown quantities from any two quadratic equations. In the preſent caſe, the equation becomes,

$$(b^2+2ax-3x^2)^2=\left(bx^2-(a-3x)(b^2+2ax-4x^2)\right)(a+b-3x),$$ or,

$$\begin{array}{l} 27x^4-30ax^3+12a^2x^2-2a^3x-a^2b^2=0 \\ \qquad -15bx^3+11abx^2+2ab^2x-ab^3 \\ \qquad\qquad\qquad -2b^2x^2-2a^2bx-b^4. \\ \qquad\qquad\qquad\qquad\quad +3b^3x. \end{array}$$

The ſame method, it is evident, may be applied, whatever be the number, or the order of the equations.

4. By a ſimilar proceſs, the radical quantities that enter into any equation may be exterminated. This cannot always be effected by involution only; for, on the contrary, by that operation, the number of radicals may ſometimes be increaſed. The method, on the other hand, which we are going to lay down, is univerſal:

It conſiſts in making each ſurd quantity equal to a new unknown quantity, and in exterminating from theſe, and from the given equation, all the unknown quantities but one, in the manner that has been explained.

Let there be given $x-\sqrt[3]{a^2x}-\sqrt{x^2-a^2}=0$, Make $\sqrt[3]{a^2x}=y$, and $\sqrt{x^2-a^2}=z$. then $x-y-z=0$, $y^3-a^2x=0$, and $z^2-x^2+a^2=0$. From theſe equations z is eaſily exterminated, by taking its value in the firſt, and ſubſtituting it in the laſt; we have then $z=x-y$, and $z^2=x^2-2yx+y^2$, ſo that $-2xy+y^2+a^2=0$, or $y^2-2xy+a^2=0$, Now, we have alſo $y^3-a^2x=0$, and it remains to exterminate y from theſe equations.

(A)	$y^2-2xy+x^2=0$
(B)	$y^3-a^2x=0$
A into y	$y^3-2xy^2+a^2y=0$
B ſubtracted	$-2xy^2+a^2y+a^2x=0$
A into $2x$	$+2xy^2-4x^2y+2a^2x=0$
Add theſe laſt, (C)	$(a^2-4x^2)y+3a^2x=0$

Therefore $y=\frac{-3a^2x}{a^2-4x^2}$.

C into y $(a^2-4x^2)y^2+3a^2xy=0$

A into (a^2-4x^2) $(a^2-4x^2)y^2-2a^2xy+8x^3y+a^4-4a^2x^2=0$

This ſubtracted, $(5a^2x-8x^3)y+4a^2x^2-a^4=0$

Therefore, $y=\frac{a^4-4a^2x^2}{5a^2x-8x^3}$

Hence, equating the values of y, we have

$$\frac{a^4-4a^2x^2}{5a^2x-8x^3}=\frac{-3a^2x}{a^2-4x^2}.$$

And $(a^2-4x^2)^2=-3x(5a^2x-8x^3)$

Or $8x^4-7a^2x^2-a^4=0$.

The given equation, when freed from radicals, is therefore no more than a quadratic.

5. It is to be obſerved concerning all theſe exterminations, that, if m and n be the orders of two equations containing x and y, the order of the equation that reſults

from them, ought not to exceed *mn*. It may be leſs, as in this example; but, if it come out greater, the equation admits of a diviſor, which, in moſt caſes, will be eaſily diſcovered.

APPENDIX.

No. IV.

Approximations.

1. SIR ISAAC NEWTON's method of approximating to the roots of equations is rendered ſomewhat more ſimple, by making all the ſubſtitutions in the ſame formula, ſo that each ſtep may be derived from the preceding, preciſely in the ſame manner. Thus, in the example of the text, page 208, viz. $x^3-2x-5=0$, if A be any approximation to the root, and c the correction to be made on it, in order to have a more accurate value of x, that is, if $A+c=x$, then

$$x^3 = A^3+3A^2c+3Ac^2+c^3,$$
$$-2x = -2A-2c,$$
$$-5 = -5.$$

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Therefore, $A^3-2A-5+3A^2c-2c+3Ac^2+c^3=0$, and rejecting the terms where c is of a higher power than the firſt, we have $A^3-2A-5+3A^2c-2c=0$, or $c=\frac{-A^3+2A+5}{3A^2-2}$.

The correction c, thus found, being added to A, or ſubtracted from it, according as it is poſitive or negative, will give a nearer value of x; and this value being in its turn ſubſtituted for A in the formula, will give a new correction, by which it will be rendered ſtill more accurate. We may continue theſe operations till any aſſigned degree of exactneſs is obtained; and it will conſiderably abridge the labour, if, in raiſing the powers, at each ſtep, we avail ourſelves of thoſe that were raiſed in the preceding.

2. The correction $c=\frac{-A^3+2A+5}{3A^2-2}$, may be more conveniently expreſſed by performing the diviſion by $3A^2-2$, in part; we have then $c=-\frac{1}{3}A+\frac{\frac{4}{3}A+5}{3A^2-2}$.

In general, if $x^3+px^2+qx+r=0$, be any cubic equation, A a near value of x, and c the correction to be applied to A, to give a value of it ſtill nearer, then

$$c=-\frac{A^3+pA^2+qA+r}{3A^2+2pA+q};$$

$$\text{or } c=-\tfrac{1}{3}A-\frac{\frac{1}{3}pA^2+\frac{2}{3}qA+r}{3A^2+2pA+q}.$$

It is evident that this method of approximating extends to equations of every order, and to thoſe with literal as well as with numeral coefficients.

3. The corrections by which we approximate to the root of an equation, form a ſeries of terms that gradually decreaſe, each of them being leſs than the preceding; and this circumſtance has ſuggeſted another method by which the terms of ſuch a ſeries may be directly inveſtigated. This is called the method of *indeterminate coefficients*, and is of very extenſive uſe in the higher parts of the mathematics. To explain the nature of it, let us ſuppoſe that the equation to be reſolved is $x^3+qx-r=0$.

It is plain that x depends on q and r only, and is therefore equal to a certain combination of the powers of theſe two quantities.

In this combination of powers, let A be the ſum of all the terms that do not involve q; Bq the ſum of all thoſe that involve only the firſt power of q; Cq^2 the ſum of all thoſe that involve only the ſecond, &c. Then $x=A+Bq+Cq^2+Dq^3+$, &c. where, according to circumſtances, the ſeries may either terminate, or continue *ad infinitum*, and where A, B, C, D, &c. are quantities that depend only on r, and are in no way related to q. Let this ſeries and its powers be ſubſtituted for x and its powers, in the given equation, we have then,

$$\left.\begin{array}{llllll}
x^3= A^3 & +3A^2Bq & +3AB^2q^2 & +3A^2Dq^3+ & 3AC^2q^4+, \&c. \\
 & & +3A^2Cq^2 & +6ABCq^3+ & 3B^2Cq^4 \\
 & & + & B^3q^3+ & 6ABDq^4 \\
 & & & + & 3A^2Eq^4 \\
+qx= \;+ & Aq & +Bq^2 & +Cq^3+ & Dq^4+, \&c. \\
-r=-r. & & & &
\end{array}\right\}=0.$$

4. Now, it is not enough that the whole amount of theſe terms, when taken together, ſhould be equal to 0, for this would

infer a certain relation between A, B, C, &c. and the quantity q, contrary to the ſuppoſition. The quantity q, therefore, muſt diſappear from the equation; and this can no otherwiſe happen, than by all the terms, where q is of the ſame dimenſion, being together equal to 0. Therefore we have $A^3 - r = 0$; $3AB + 1 = 0$; $3AB^2 + 3A^2C + B = 0$; $3A^2D + 6ABC + B^3 + C = 0$; $3AC^2 + 3B^2C + 6ABD + 3A^2E + D = 0$; Hence $A = r^{\frac{1}{3}}$; $B = -\frac{q}{3r^{\frac{1}{3}}}$; $C = 0$; $D = \frac{1}{81r^{\frac{5}{3}}}$; $E = -\frac{1}{243r^{\frac{7}{3}}}$, &c. the coefficients A, B, C, D, &c. are thus determined in terms of r, as they ought to be, and ſo the ſeries,

$$A + Bq + Cq^2 + Dq^3 + Eq^4 +, \text{ or}$$

$$x = r^{\frac{1}{3}} - \frac{q}{3r^{\frac{1}{3}}} + \frac{q^3}{81r^{\frac{5}{3}}} - \frac{q^4}{243r^{\frac{7}{3}}} +, \&c.$$

5. A few terms of this ſeries will give the value of x very near the truth, provided q be a ſmall quantity, in reſpect of r, and each term of conſequence much leſs

than the preceding. A ſeries is ſaid to *converge* when its terms decreaſe in this manner; and it is evident that this ſeries will converge the faſter, the leſs that q is in reſpect of r.

6. As in the equation $x^3+qx-r=0$, x muſt depend on the two quantities q and r, we may either ſuppoſe x equal to a ſeries $A+Bq+$, &c. as has been done; or to a ſeries $A+Br+Cr^2+Dr^3+$, &c. where A, B, C, D, are quantities that involve only q, and are to be determined as in the laſt example. But as, on trial, we would find, in this inſtance, the coefficients of all the even powers of r to vaniſh, we may aſſume the ſeries, ſuch as to involve only the odd powers of r, ſo that $x=Ar+Br^3+Cr^5+Dr^7$, &c. Then, by ſubſtitution,

$$\left.\begin{array}{llllll}
x^3= & & A^3r^3 & +3A^2Br^5 & +3AB^2r^7 & +3A^2Dr^9+,\ \&c. \\
 & & & & +3A^2Cr^7 & +6ABCr^9 \\
 & & & & & +\quad B^3r^9 \\
+qx= & Aqr & +Bqr^3+ & Cqr^5+ & Dqr^7 & +\quad Eqr^9+,\ \&c. \\
-r= & -r. & & & &
\end{array}\right\}=0$$

Therefore, equating the coefficients of the terms where r is of the ſame dimenſion, $Aq-1=0$; $A^3+Bq=0$; $3AB^2+3A^2C+Dq=0$, &c. Hence $A=\frac{1}{q}$, $B=-\frac{1}{q^4}$, $C=\frac{3}{q^7}$; $D=-\frac{12}{q^{10}}$, &c.; and therefore $x=\frac{r}{q}-\frac{r^3}{q^4}+\frac{3r^5}{q^7}-\frac{12r^7}{q^{10}}$, &c.

This ſeries will converge the more rapidly the leſs that r is in reſpect of q, and therefore will become uſeful juſt when the other ceaſes to be ſo.

7. In inveſtigations of this ſort, a difficulty ariſes, both with reſpect to the firſt term, and the progreſſion of the exponents, in the ſeries to be aſſumed. But this will be avoided, if we aſſume $x=Aq^m+Bq^n$, (q being the quantity according to the powers of which the ſeries is to proceed), and ſubſtitute theſe two terms for x in the given equation. It will then be eaſily diſcovered what values muſt be aſſigned to m and n, in order that the equating of the coefficients, of the like powers of q, may

ſucceed. When m and n are found, the initial terms of the ſeries, and the progreſſion of the exponents are both determined.

8. By the ſame method of aſſuming a ſeries with indeterminate coefficients, even the more common operations may be facilitated; evolution, for inſtance, may be converted into involution, and diviſion into multiplication.

Let it be required to extract the ſquare root of $a^2 - x^2$; aſſume $\sqrt{a^2 - x^2} = A + Bx^2 + Cx^4 + Dx^6 +$, &c. then ſquaring both ſides, and tranſpoſing,

$$0 = \begin{cases} A^2 + 2ABx^2 + 2ACx^4 + 2ADx^6 +, \text{ \&c.} \\ \qquad + x^2 + B^2x^4 + 2BCx^6 +, \text{ \&c.} \\ -a^2 \end{cases}$$

Hence $A^2 - a^2 = 0$; $2AB + 1 = 0$; $2AC + B^2 = 0$; $AD + BC = 0$, &c. From theſe equations we have $A = a$, $B = -\frac{1}{2a}$, $C = -\frac{1}{8a^3}$, $D = -\frac{1}{16a^5}$, &c. Therefore $\sqrt{a^2 - x^2} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$, &c.

9. As an example of the application of the ſame method to the finding of a quotient, let there be given the fraction $\frac{1}{a+bx+dx^2}$, and let $\frac{1}{a+bx+dx^2} = A + Bx + Cx^2 + Dx^3 +$, &c.

Then multiplying by $a+bx+dx^2$, and tranſpoſing all the terms to one ſide of the equation, we have,

$$0 = \left\{ \begin{array}{l} aA + aBx + aCx^2 + aDx^3 +, \text{ \&c.} \\ \qquad Abx + Bbx^2 + Cbx^3 +, \text{ \&c.} \\ \qquad\qquad Adx^2 + Bdx^3 +, \text{ \&c.} \\ -1. \end{array} \right.$$

Hence $aA - 1 = 0$, $aB + Ab = 0$, $aC + Bb + Ad = 0$, $aD + Cb + Bd = 0$; that is, $A = \frac{1}{a}$, $B = -\frac{b}{a^2}$, $C = \frac{b^2 - ad}{a^3}$, &c. Here the law of continuation is evident, the coefficient of each term being derived in the ſame manner, from thoſe of the two preceding terms.

In caſes where both diviſion and evolution are to be performed at the ſame time, as in the expreſſion $\frac{1}{\sqrt{a+bx}}$, or $\frac{\sqrt{a^2-x^2}}{\sqrt[3]{a^3+x^3}}$, this method will be found more expeditious than any other.

APPENDIX.

No. V.

Of Logarithms.

ALL affirmative numbers may be conſidered as powers of any one given affirmative number. The powers of 2, for inſtance, may ſucceſſively become equal, either exactly, or within leſs than any aſſigned difference, to all numbers whatſoever, from 0 upwards. Thoſe powers which have integers for their exponents, viz. 2^0, 2^1, 2^2, 2^3, 2^4, &c. give the geometrical progreſſion 1, 2, 4, 8, 16, 32, &c. and the intermediate numbers are expreſſed, at leaſt nearly, by powers of 2 having fractional exponents. Thus, $2^{1.58496} = 3$ nearly; $2^{2.3219} = 5$, $2^{2.58496} = 6$, &c.

In like manner, might the powers of 10 be taken to expreſs all numbers; thus, $10^{0}=1$, $10^{.30103}=2$, $10^{.47712}=3$, &c. Fractions alſo may be thus expreſſed, for $10^{-1}=\frac{1}{10}$, $10^{-.30103}=\frac{1}{2}$, &c. the negative powers of 10 being now employed.

In the ſame manner, any other number, even a fraction, might be taken, in place of 2 or 10, in the preceding examples, and ſuch exponents might be found as would give its powers equal to all numbers, from 0 upwards.

There are, therefore, no limitations with reſpect to the magnitude of the number, of which the powers are to repreſent all other numbers, except that it muſt neither be equal to unity, nor negative. If it be $=1$, then all its powers will alſo be $=1$; and if it be negative, there will be numbers to which none of its powers can poſſibly be equal.

2. If, therefore, y denote any number whatever, and a a given number, n may be found ſuch that $a^{n}=y$; and n, that is, *the*

exponent of a, *which gives a power equal to* y, *is called the logarithm of* y. In the common tables of logarithms, $a=10$.

3. Hence the properties of logarithms are eaſily deduced.

The ſum of the logarithms of any two numbers is equal to the logarithm of their product. Let b and c be the numbers, m and n their logarithms, ſo that $a^m=b$, and $a^n=c$, then $a^m \times a^n = bc$, or $a^{m+n} = bc$; $m+n$, that is, the ſum of the logarithms of b and c, is therefore, by the definition, the logarithm of bc.

4. The difference of the logarithms of two numbers is equal to the logarithm of their quotient. For, if $a^m=b$, and $a^n=c$, $\frac{a^m}{a^n} = \frac{b}{c}$, that is, $a^{m-n} = \frac{b}{c}$; $m-n$, or the difference of the logarithms of b and c, is therefore the logarithm of $\frac{b}{c}$.

5. Hence log. $b^n = n$ log. b. For b^n is b multiplied into itſelf, n number of times;

and therefore the log. of b^n is equal to the log. of b added to itſelf n number of times, or to n log. b. This is true, whether n be an integer or a fraction.

6. The logarithm of any ſingle term whatever may be aſſigned by theſe rules. Thus, log. $\frac{\sqrt{3} \times a^2 b}{2c^3} = \frac{1}{2}$ log. 3 + 2 log. a + log. b — log. 2 — 3 log. c.

7. There are different ſyſtems of logarithms, according to the radical number employed. Thus, if 10 be taken for the radical number, or that whereof the powers are to become ſucceſſively equal to all numbers, we have the common or tabular logarithms. In the logarithms firſt conſtructed by Lord Napier, or in what are called *hyperbolic logarithms*, the radical number is 2.7182818+, &c.

In different ſyſtems, the logarithms of the ſame number are always in a conſtant ratio to one another. Suppoſe that, in one ſyſtem, $a = b$, and in another, $r^m = b$, whatever b be, n will be to m in the ſame ratio.

For $a^n = b = r^m$, therefore $a^{\frac{n}{m}} = r$. The fraction $\frac{n}{m}$ therefore depends only on the quantities a and r, and muſt be the ſame, whatever be the value of b.

Thus, the tabular logarithm of any number is to the hyperbolic, as the tab. log. of 10 to the hyperbolic logarithm of 10, that is, as 1 to 2.30258509+, &c. The logarithms of any one ſyſtem, therefore, being computed, thoſe of any other may eaſily be found. In all ſyſtems, the logarithm of 1 is 0.

8. When the logarithm of any number is given, that of a number, a little greater or leſs than it may be found by the binomial theorem. If, for inſtance, we have the logarithm of 1000, which in the common tables is 3, we can eaſily find that of 1001. For, ſince $1001 = 1000 + 1 = 1000 (1 + \frac{1}{1000})$, therefore log. 1001 = log. 1000 + log. $(1 + \frac{1}{1000}) = 3 +$ log. $(1 + \frac{1}{1000})$. The computation of the logarithm of 1001 is therefore reduced to that of $1 + \frac{1}{1000}$; and, in general, the

computation of the logarithm of a number, not very different from one of which the logarithm is given, is reduced to that of the logarithm of $1+x$, x being a ſmall fraction.

Let the logarithm of $1+x$ be λ, the radical number, that is 10, being $=1+b$. Then $1+x=(1+b)^{\lambda}$, or raiſing both ſides of this equation to the power λ, $(1+x)^{\lambda}=(1+b)^{\lambda\lambda}$, and reſolving both into ſeries, by the binomial theorem,

$1+\lambda x+\lambda.\frac{\lambda-1}{2}x^2+\lambda.\frac{\lambda-1}{2}.\frac{\lambda-2}{3}x^3+$, &c. $=$ $1+\lambda\lambda b+\lambda\lambda.\frac{\lambda\lambda-1}{2}b^2+\lambda\lambda.\frac{\lambda\lambda-1}{2}.\frac{\lambda\lambda-2}{3}b^3+$, &c. Therefore, taking 1 from both ſides, and dividing by λ, $x+\frac{\lambda-1}{2}x^2+\frac{\lambda-1}{2}.\frac{\lambda-2}{3}x^3$ $+$, &c. $=\lambda b+\lambda.\frac{\lambda\lambda-1}{2}b^2+\lambda.\frac{\lambda\lambda-1}{2}.\frac{\lambda\lambda-2}{3}b^3$ $+$, &c.

Now, if λ is a very ſmall fraction, as it muſt be, on the ſuppoſition that x is ſo, we have $\lambda-1=-1$, nearly; $\lambda-2=-2$, nearly, &c. Therefore the equation be-

comes, $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$, &c. $= \lambda$ $(b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \frac{1}{4}b^4 +$, &c.), and $\lambda = \frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +, \&c.}{b - \frac{1}{2}b + \frac{1}{3}b^3 - \frac{1}{4}b^4 +, \&c.}$

9. The ſeries in the numerator of this fraction converges, becauſe x is ſmall; but the ſeries in the denominator, where $b = 9$, diverges, each term of it being greater than the preceding. In order that, if poſſible, it may be transformed into another, which, by converging, may afford an approximation to the truth, let its value $= m$, then $\lambda =$ log. $(1 + x) = \frac{1}{m}(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$, &c.). Now, if x be negative, log.$(1 - x) = \frac{1}{m}(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c.). Therefore log. $(1 + x) -$ log. $(1 - x) = \frac{2}{m}$ $(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 +$, &c).; that is, log. $\frac{1+x}{1-x}$ $= \frac{2}{m}(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 +$, &c.).

Let $\frac{1+x}{1-x} = 10$, then log. $\frac{1+x}{1-x} =$ log. $10 = 1$, and alſo $x = \frac{9}{11}$, therefore $1 = \frac{2}{m}$ $(\frac{9}{11} + \frac{1}{3} \times \frac{9^3}{11^3} + \frac{1}{5} \times \frac{9^5}{11^5}, +$, &c.), or $m = 2\ (\frac{9}{11} + \frac{9^3}{3.11^3} + \frac{9^5}{5.11^5}, +$, &c.). This ſeries converges, and its value, that is, m is found $=2.30258509$, &c. Having thus obtained a value of m, it is evident that λ or log. $1+x$, x being any ſmall fraction, is to be found from the ſeries $\frac{1}{m}(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$, &c.).

10. When, by help of this ſeries, or others of the ſame kind, the logarithms of the prime numbers are computed, thoſe of all other numbers may be deduced from them. Thus, if the logarithms of 2 and 3 are known, we have from thence the logarithms of an infinity of other numbers; log. 4 = 2 log. 2; log. 6 = log. 2 + log. 3; log. 5 = log. 10 − log. 2 = 1 − log. 2; log. 8 = 3 log. 2; log. 9 = 2 log. 3; log. 12 = 2 log. 2 + log. 3, &c.; alſo log. 20 = 1 + log. 2, log. 25 = 2 log. 5, log. 30 = 1 + log. 3, &c.

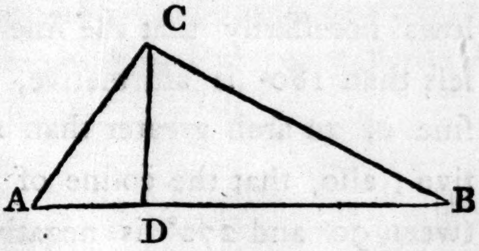
APPENDIX.

No. VI.

Arithmetic of Sines, &c.

1. THE application of algebra to trigonometry has given riſe to a calculus of a particular kind, which is of great uſe in the higher geometry. The following theorem is the foundation of it.

I. Let a and b be any two arches of a circle, of which the radius $=1$, ſin. $(a+b) =$ ſin. $a \times$ cos. $b +$ cos. $a \times$ ſin. b.



In the triangle CAB let A and B be the given angles, then ſin. ACB $=$ ſin. $(a+b)$. Draw CD perpendicular to AB. Then, becauſe

$AC : AB :: \text{ſin. } b : \text{ſin. } a+b$; and $\text{ſin. } b = \frac{CD}{CB}$, $AC : AB :: \frac{CD}{CB} : \text{ſin.} a+b = \frac{AB \times CD}{AC \times CB} = \frac{(AD+DB)CD}{AC \times CB} = \frac{AD \times CD}{AC \times CB} + \frac{DB \times CD}{AC \times CB}$. Now, $\frac{AD}{AC} = \text{cos. } a$; and $\frac{CD}{CB} = \text{ſin. } b$; likewiſe $\frac{CD}{CA} = \text{ſin. } a$, and $\frac{DB}{BC} = \text{cos. } b$. Therefore $\text{ſin. } a + b = \text{ſin. } a \times \text{cos. } b + \text{cos. } a \times \text{ſin. } b$. Q. E. D.

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In this theorem, and in thoſe which follow, it is ſuppoſed that the ſine and coſine of an arch leſs than 90° are both affirmative. Theſe two ſuppoſitions are arbitrary; but, when they are once laid down, it follows neceſſarily that the ſine of any arch leſs than 180° is affirmative, but that the ſine of an arch greater than 180° is negative; alſo, that the coſine of an arch between 90° and 270° is negative, but that of an arch between 270° and 360°, the coſine is affirmative. Laſtly, that, when an arch changes from + to −, or from − to +, its ſine undergoes a like change, but its coſine retains the ſame ſign.

II. If, in the preceding theorem, we ſuppoſe the arch b to become negative, then ſin. b will alſo be negative, and therefore $\text{ſin.}\,a - b = \text{ſin.}\,a \times \text{cos.}\,b - \text{cos.}\,a \times \text{ſin.}\,b$.

III. $\text{Cos.}\,a+b = \text{cos.}\,a \times \text{cos.}\,b - \text{ſin.}\,a \times \text{ſin.}\,b$. For, $\text{cos.}\,a + b = \text{ſin.}\,(90^\circ - a - b) = \text{ſin.}\,c - b$, making $90^\circ - a = c$. But $\text{ſin.}\,c - b = \text{ſin.}\,c \times \text{cos.}\,b - \text{cos.}\,c \times \text{ſin.}\,b$; now, $\text{ſin.}\,c = \text{cos.}\,a$, and $\text{cos.}\,c = \text{ſin.}\,a$, therefore $\text{ſin.}\,c - b$, that is, $\text{cos.}\,a + b = \text{cos.}\,a \times \text{cos.}\,b - \text{ſin.}\,a \times \text{ſin.}\,b$.

IV. If, in the laſt theorem, we make b negative, ſin. b will alſo become negative, and therefore $\text{cos.}\,a - b = \text{cos.}\,a \times \text{cos.}\,b + \text{ſin.}\,a \times \text{ſin.}\,b$.

2. From theſe theorems we derive the following.

V. Since ſin. $a + b =$ ſin. $a \times$ cos. $b +$ cos. $a \times$ ſin. b, and ſin. $a - b =$ ſin. $a \times$ cos. $b -$ cos. $a \times$ ſin. b; therefore ſin. $a + b +$ ſin. $a - b = 2$ ſin. $a \times$ cos. b.

VI. By ſubtracting the ſame two equations,

Sin. $a + b -$ ſin. $a - b = 2$ cos. $a \times$ ſin. b.

VII. Alſo, ſince by theor. 3d, cos. $a + b =$ cos. $a \times$ cos. $b -$ ſin. $a \times$ ſin. b, and, by theor. 4. cos. $a - b =$ cos. $a \times$ cos. $b +$ ſin. $a \times$ ſin. b, therefore cos. $a + b +$ cos. $a - b = 2$ cos. $a \times$ cos. b.

VIII. Again, by ſubtracting, cos. $a + b -$ cos. $a - b = - 2$ ſin. $a \times$ ſin. b, or cos. $a - b -$ cos. $a + b = 2$ ſin. $a \times$ ſin. b.

3. By help of theſe theorems, the products and powers of the ſines and coſines of arches may be expreſſed in terms of the ſums or differences of the ſines or coſines of certain multiples of thoſe arches. Thus, ſince 2 ſin. $a \times$ ſin. $b =$ cos. $a - b -$ cos. $a + b$, if $b = a$, 2 ſin. $^2 a = 1 -$ cos. $2a$.

Again, ſin. $^3a = \frac{1}{2}$ (1 — cos. 2 a) ſin. a, but cos. 2 a × ſin.a = $\frac{1}{2}$ (ſin. 3 a — ſin. a), ſo that ſin. ^3a = $\frac{1}{2}$ (ſin. a + $\frac{1}{2}$ (ſin. a — ſin. 3a)), or 4 ſin. ^3a = 2 ſin. a + ſin. a — ſin. 3a, that is, 4 ſin. ^3a = 3 ſin. a. — ſin. 3a.

Thus, we obtain the values of the powers of ſin. a, as in this table :

ſin. ^1a = ſin. a

2 ſin. ^2a = 1 — cos. 2a

4 ſin. ^3a = 3 ſin. a — ſin. 3a

8 ſin. ^4a = 3 — 4 cos. 2 a + cos. 4a

16 ſin. ^5a = 10 ſin. a — 5 ſin. 3 a + ſin. 5 a.

&c.

In the ſame manner, it is found that,

cos. ^1a = cos. a

2 cos. ^2a = 1 + cos. 2 a

4 cos. ^3a = 3 cos. a + cos. 3a

8 cos. ^4a = 3 + 4 cos. 2 a + cos. 4 a

16 cos. ^5a = 10 cos. a + 5 cos. 3 a + cos. 5 a.

&c.

4. Again, converſely, the ſine or coſine of any arch may be expreſſed in the powers of an arch, whereof it is a multiple. Thus, by Theor. 1ſt, ſin. $a + b$ = ſin. a × cos. b

$+$ cos. $a \times$ ſin. b, and of $a = b$, ſin. $2a$ $=$ ſin. $a \times$ cos. $a +$ cos. $a \times$ ſin. a $= 2$ ſin. $a \times$ cos. a.

Alſo, ſin. $2a + a =$ ſin. $2a \times$ cos. a $+$ cos. $2a \times$ ſin. a. Now, ſin. $2a =$ 2 ſin. $a \times$ cos. a, and cos. $2a = 2$ cos.^2a $- 1$. Wherefore, ſin. $3a = 2$ ſin. $a \times$ cos. $^2a +$ ſin. a $(2$ cos.$^2a - 1) = 2$ ſin. a $(1 -$ ſin. $^2a) +$ ſin. a $(2 - 2$ ſin. $a^2 - 1)$ $= 2$ ſin. $a - 2$ ſin. $^3a + 2$ ſin. $a -$ 2 ſin.$^3a -$ ſin. $a = 3$ ſin. $a - 4$ ſin. ^3a.

Thus there is formed the following Table :

ſin. $a =$ ſin. a

ſin. $2a = 2$ ſin. $a \times$ cos. a

ſin. $3a = 3$ ſin. $a - 4$ ſin. ^3a

ſin. $4a = (4$ſin. $a - 8$ ſin. $^3a)$ cos. a,

&c.

Alſo, cos. $a =$ cos. a

cos. $2a = 2$ cos. $^2a - 1$

cos. $3a = 4$ cos. $^3a - 3$ cos. a

cos. $4a = 8$ cos. $^4a - 8$ cos. $^2a + 1$,

&c.

5. From one of theſe theorems may be derived the ſolution of thoſe cubic equations

which cannot be reſolved by Cardan's rule. As we have cos. $3a = 4\cos.^3 a - 3\cos. a$, if we put cos. $a = x$, we have $4x^3 - 3x -$ cos. $3a = 0$, or $x^3 - \frac{3}{4}x - \frac{1}{4}$ cos. $3a = 0$; where, if $3a$ be a given arch, x is the coſine of the 3d part of that arch, and is found by the triſection of it. To render this equation more general, let the root, or x, be multiplied by m, a determinate but unknown quantity, then $x^3 - \frac{3}{4}m^2x - \frac{m^3}{4}$ cos. $3a = 0$, and the roots of this equation are ſtill found by triſecting the arch $3a$, and multiplying the coſine of a by m.

Now, let $x^3 - qx - r = 0$, repreſent any cubic equation; then, if we ſuppoſe $q = \frac{3}{4}m^2$, and $r = \frac{m^3}{4}$ cos. $3a$, $m = \frac{2\sqrt{q}}{\sqrt{3}}$, and cos. $3a = \frac{4r}{m^3} = \frac{3r\sqrt{3}}{2q\sqrt{q}}$. An arch, therefore, muſt be found, of which the coſine is $\frac{3r\sqrt{3}}{2q\sqrt{q}}$, and the coſine of the third part of this arch, when multiplied by m, or by $\frac{2\sqrt{q}}{\sqrt{3}}$, is the root required. The cubic equations that can be reſolved in this manner are thoſe, in which $\frac{3r\sqrt{3}}{2q^2\sqrt{q}}$ being leſs than 1, can repreſent the coſine of an arch, and are exactly the ſame that fall under the irreducible caſe of Cardan's rule.

F I N I S.

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